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# SENSITIVITY ANALYSIS OF MULTIPLE EIGENVALUES AND ASSOCIATED EIGENVECTORS OF QUADRATIC EIGENPROBLEM[†]

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The article is focused on the sensitivity of eigenvalues and eigenvectors in a quadratic eigenvalue problem with real matrices defining the problem under consideration and under the strong assumption that these matrices form a non-defective operator. The particular interest is the case of multiple eigenvalues and associated eigenvectors. Generally in such a case derivatives in the Fréchet sense do not exist, but only in the Gâteaux sense. The formulas of the directional differential in the closed matrix form were derived. A numerical example is shown.

**Keywords:** quadratic eigenvalue problem; multiple eigenvalues; sensitivity analysis.



## 1. Introduction

Conscious shaping of the dynamic characteristics of a structure is an essential duty of the designer, especially for structures where the dynamic phenomena require non-standard calculations and not just the use of dynamic coefficients as static load multipliers. Knowledge of the properties of the eigenproblem and its importance in the dynamics of the whole system is fundamental in such cases. Determining the sensitivity of the eigenvalues and eigenvectors can be helpful in structural optimization or identification. On the basis of the presented formulas, an original programme was developed that enables the calculation of the directional derivatives of eigenvalues and eigenvectors.

Sensitivity analysis related to the eigenproblem has been intensively developed for at least 80 years. On both the mathematical and numerical side, it is still a research problem – especially in terms of the numerical efficiency of the algorithms developed (Łasecka-Plura, 2023; 2024; Phuor & Yoon, 2023; Martinez-Agirre & Elejabarrieta, 2011; Wang & Dai, 2015). Indeed, the mathematical foundations and basic physical interpretations of the problems of sensitivity analysis of multiple eigenvalues and eigenvectors have been well known since the 1990s. In general, the majority of papers deals with a symmetric problem, i.e., when the matrices of a system of equations of motion in the configuration space are symmetric.

The linear eigenproblem (LEP) $(\mathbf{A}-\mathbf{I})\,\mathbf{I}=\mathbf{0}$, is extensively discussed in abundant mathematical literature, as well as in the literature on dynamical systems (Horn & Johnson, 2013; Garcia & Horn, 2017). The quadratic eigenproblem (QEP) is less frequently discussed due to the computational practice of reducing a QEP to a LEP via an isospectral transformation (Xu & Wu, 2008). It should be noted that QEPs and LEPs are nevertheless different because of the different spaces of the eigenvectors and their properties (Tisseur & Meerbergen, 2001; Lancaster & Zaballa, 2009; Lancaster, 2013).

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The method for deriving the derivatives presented in this paper combines and extends various concepts presented in (Seyranian *et al.*, 1994; Krog & Olhoff, 1995; Lee *et al.*, 1999b) – the combining element is the way in which the eigenvalues are numbered and ordered, and the extension pertains to the non-symmetry of matrices in the system of the equations of motion in configuration space.

It will be assumed in the rest of the text that $\mathbf{h} = \left[h_1, ..., h_{N_p}\right]$ denotes a parameter vector in the design parameter space, $\mathbf{h} \in \mathcal{R}^{N_p}$, $N_p$ – dimension of the space. In that space, the directional versor $\mathbf{e} \in \mathcal{R}^{N_p}$, ($\|\mathbf{e}\| = 1$) is also defined. All quantities appearing in the QEP, therefore, remain dependent on the vector $\mathbf{h}$.

For convenience, let us introduce, according to (Andrew *et al.*, 1993; Lancaster, 2013; Lancaster & Zaballa, 2009), the operator $\mathbf{L}$ defined as follows:

$$\mathbf{L}\left(\lambda(\mathbf{h}), \mathbf{h}\right) = \lambda^2(\mathbf{h})\mathbf{M}(\mathbf{h}) + \lambda(\mathbf{h})\mathbf{C}(\mathbf{h}) + \mathbf{K}(\mathbf{h}), \tag{1.1}$$

where the dependence on the parameter vector is explicitly indicated. Then the QEP can be written in abbreviated form:

$$\mathbf{L}\left(\lambda(\mathbf{h}), \mathbf{h}\right)\mathbf{\Psi}(\mathbf{h}) = \mathbf{0}, \qquad \mathbf{\Phi}^H(\mathbf{h})\mathbf{L}\left(\lambda(\mathbf{h}), \mathbf{h}\right) = \mathbf{0}. \tag{1.2}$$

It follows that both eigenvalues and eigenvectors are mappings of the vector $\mathbf{h}$, i.e., $\lambda = \lambda(\mathbf{h})$, $\mathbf{\Psi} = \mathbf{\Psi}(\mathbf{h})$, and $\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{h})$ – in the general case, these mappings for multiple eigenvalues are no longer differentiable in the Fréchet sense, but only in the Gâteaux one.

It is assumed that the matrices occurring in the eigenproblem are dependent on the vector $\mathbf{h}$, i.e.:

$$\mathbf{M} = \mathbf{M}(\mathbf{h}), \qquad \mathbf{C} = \mathbf{C}(\mathbf{h}), \qquad \mathbf{K} = \mathbf{K}(\mathbf{h}), \tag{1.3}$$

and it is assumed also that these matrices are differentiable in the Fréchet sense, so it implies the existence of partial derivatives and the Taylor series expansion in the form:

$$\begin{aligned}
\mathbf{K}(\mathbf{h} + \epsilon\mathbf{e}) &= \mathbf{K}(\mathbf{h}) + \epsilon\sum_{p=1}^{N_p}\frac{\partial\mathbf{K}(\mathbf{h})}{\partial h_p}e_p, \\
\mathbf{C}(\mathbf{h} + \epsilon\mathbf{e}) &= \mathbf{C}(\mathbf{h}) + \epsilon\sum_{p=1}^{N_p}\frac{\partial\mathbf{C}(\mathbf{h})}{\partial h_p}e_p, \\
\mathbf{M}(\mathbf{h} + \epsilon\mathbf{e}) &= \mathbf{M}(\mathbf{h}) + \epsilon\sum_{p=1}^{N_p}\frac{\partial\mathbf{M}(\mathbf{h})}{\partial h_p}e_p,
\end{aligned} \tag{1.4}$$

where $\epsilon \in \mathcal{R}$.

The article is focused on the sensitivity of eigenvalues and eigenvectors in a QEP with real matrices defining the problem under consideration and under the strong assumption that these matrices form the non-defective operator $\mathbf{L}$ (i.e., the operator $\mathbf{L}$ is diagonalizable). The QEP is defined in classical form by the equation:

$$\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\right)\mathbf{\Psi} = \mathbf{0}, \qquad \mathbf{\Phi}^H\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\right) = \mathbf{0}. \tag{1.5}$$

In Eqs. (1.5): $\mathbf{M}^T \neq \mathbf{M}$, $\mathbf{C}^T \neq \mathbf{C}$, $\mathbf{K}^T \neq \mathbf{K}$, $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathcal{R}^{N\times N}$ and $\lambda_i \in \mathcal{C}$ ($\lambda_i = \sigma_i + \omega_i\mathrm{i}$, $\lambda_i^* = \sigma_i - \omega_i\mathrm{i}$, $\mathrm{i}^2 = -1$, $\sigma_i, \omega_i \in \mathcal{R}$), while for eigenvectors: $\mathbf{\Psi}_i, \mathbf{\Phi}_i \in \mathcal{C}^N$ with $i = 1, ..., 2N$.

As can be seen, an asymmetry of the matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ is assumed here – such cases are unusual and rare in the practice of computation and analysis of various technical problems. The asymmetry of the $\mathbf{K}$ matrix appears in the presence of follower forces, the asymmetry of the $\mathbf{M}$

matrix appears in the presence of hydrodynamic forces (i.e., in the flow of fluid around bodies), while the asymmetry of the $\mathbf{C}$ matrix arises in the presence of gyroscopic and/or Coriolis forces.

In the eigenproblem given by Eqs. (1.5), left and right eigenvectors are considered – both the eigenvalues and the corresponding eigenvectors are either complex or real, and, furthermore, the left eigenvector $\mathbf{\Phi}$ and the right eigenvector $\mathbf{\Psi}$ are different. An overview of the formulation QEP with a discussion of its properties can be found in (Tisseur & Meerbergen, 2001; Day & Walsh, 2007). It should be emphasised that Eqs. (1.5) do not represent a classically understood eigenproblem in the sense that the eigenvectors are complex – which causes the vector components to differ in phase with respect to each other. From a physical point of view, this means that standing waves will not appear in the system (as for a classically understood system without damping or with proportional damping), but travelling waves will, because the nodes and antinodes of the vibrational modes do not have a fixed position – they are variable in space and time.

In general, the issue presented in the paper with the non-symmetry of all matrices simultaneously is rarely encountered in engineering practices. In civil engineering problems, asymmetry mainly occurs with follower loads. However, even with symmetric matrices, the sensitivity analysis of multiple eigenvalues presents difficulties because the multiple eigenvalue is always differentiable in the Gâteaux sense, but is not always differentiable in the Fréchet sense.

## 2. Basic properties of QEP

Equations (1.5) have the so-called trivial (obvious) solutions at $\mathbf{\Psi} = \mathbf{0}$ and $\mathbf{\Phi} = \mathbf{0}$. On the other hand, non-trivial solutions called eigenvectors are obtained when one puts into the $\lambda$ the solutions of the characteristic equation $W(\lambda)$, i.e., the roots of the polynomial resulting from the expansion of the determinant:

$$W\left(\lambda(\mathbf{h}), \mathbf{h}\right) \equiv \det\left(\mathbf{L}\left(\lambda(\mathbf{h}), \mathbf{h}\right)\right) = 0. \tag{2.1}$$

These solutions (roots) of Eq. (2.1) are called eigenvalues, which can be single (simple) or multiple.

An important property of the characteristic polynomial (Eq. (2.1)) is its differentiability in the sense of Gâteaux at every point $\mathbf{h}$ and in every direction $\mathbf{e}$, while in the Fréchet sense – only beyond the multiple eigenvalues (Balakrishnan, 1976; Tisseur & Meerbergen, 2001; Gekeler, 2008; Seyranian *et al.*, 1994).

The Gâteaux differential of the mapping $\mathbf{g}\left(\mathbf{h}\right) : \mathcal{R}^{N_p} \to \mathcal{R}^n$ at the point $\mathbf{h}$ and in the direction $\mathbf{e}$ is called the mapping $d\widetilde{\mathfrak{G}}\left(\mathbf{h}; \epsilon\mathbf{e}\right)$ such that:

$$\forall_{\epsilon\in\mathcal{R}_+} \quad d\widetilde{\mathfrak{G}}(\mathbf{h};\mathbf{e}) = \lim_{\epsilon\to 0^+} \frac{\mathbf{g}(\mathbf{h}+\epsilon\mathbf{e}) - \mathbf{g}(\mathbf{h})}{\epsilon} \equiv \frac{d}{d\epsilon}\mathbf{g}(\mathbf{h}+\epsilon\mathbf{e})\bigg|_{\epsilon=0} \tag{2.2}$$

under the condition that this limit exists (Gekeler, 2008). If this limit exists, then it (thus the differential $d\widetilde{\mathfrak{G}}$) is determined uniquely. The mapping $d\widetilde{\mathfrak{G}}$ is an element of the space $\mathcal{R}^n$. Equation (2.2) gives a way of calculating the Gâteaux differential – either directly from the limit definition or as the derivative of a function of one variable $\epsilon$. The Gâteaux derivative is always homogeneous

$$\forall_{\alpha\in\mathcal{R}_+} \quad d\widetilde{\mathfrak{G}}(\mathbf{h};\alpha\mathbf{e}) = \alpha d\widetilde{\mathfrak{G}}(\mathbf{h};\mathbf{e}), \tag{2.3}$$

but not always additive, so in general

$$d\widetilde{\mathfrak{G}}(\mathbf{h};\mathbf{e}_1+\mathbf{e}_2) \neq d\widetilde{\mathfrak{G}}(\mathbf{h};\mathbf{e}_1) + d\widetilde{\mathfrak{G}}(\mathbf{h};\mathbf{e}_2). \tag{2.4}$$

The lack of additivity means that the Gâteaux directional derivative is not always a linear operator as the Fréchet operator is.

If the inertia matrix is non-singular $\det(\mathbf{M}) \neq 0$, then the operator $\mathbf{L}$ is regular and the characteristic polynomial has $2N$ finite solutions (Tisseur & Meerbergen, 2001). The set of different roots of the polynomial $W(\lambda)$ is called the spectrum $\widetilde{\mathcal{S}}_{\mathbf{L}}$ of the operator $\mathbf{L}$:

$$\widetilde{\mathcal{S}}_{\mathbf{L}} = \{\lambda_m \in \mathcal{C} \ : \ W(\lambda_m) = 0\}, \qquad m = 1, ..., \widetilde{\Omega}, \tag{2.5}$$

where $\widetilde{\Omega}$ is the number of different roots in the spectrum. Let us mark an important feature of the spectrum $\widetilde{\mathcal{S}}_{\mathbf{L}}$: when the matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are arbitrary, but real, or are complex, but Hermitian, the spectrum $\widetilde{\mathcal{S}}_{\mathbf{L}}$ is symmetric against the real axis in the complex plane, i.e., the elements of the spectrum appear as real numbers $\lambda \in \mathcal{R}$ or as coupled roots $(\lambda, \lambda^*)$.

The concept of the spectrum $\widetilde{\mathcal{S}}_{\mathbf{L}}$, in particular the position of the eigenvalues on the complex plane, plays a fundamental role in the study of mechanical systems, due to the fact that the solution of the eigenproblem is the basis for determining the motion of the system without external forces and excited only by the initial conditions. Such a motion of a mechanical system reveals its inherent properties depending only on the boundary conditions, the distribution of stiffness, the distribution of masses and damping, the susceptibility of the connections, the materials used, the dissipative properties, etc.

In general, among the solutions of Eq. (2.1), there may be real elements $\lambda_i = \sigma_i \pm \mathrm{i}0$ and purely imaginary elements $\lambda_i = 0 \pm \mathrm{i}\omega_i$ and complex elements $\lambda_i = \sigma_i \pm \mathrm{i}\omega_i$. For the assumption of real matrices, the complex elements, if present, are always paired with their conjugate (there is therefore always an even number of them). It is therefore possible (under the assumptions made), without loss of generality, to consider narrowing the spectrum $\widetilde{\mathcal{S}}_{\mathbf{L}}$ to a subset $\mathcal{S}_{\mathbf{L}} \subseteq \widetilde{\mathcal{S}}_{\mathbf{L}}$ containing all real eigenvalues (if existent) and only complex eigenvalues (if existent) with, e.g., negative imaginary parts:

$$\mathcal{S}_{\mathbf{L}} = \Big\{\lambda \in \widetilde{\mathcal{S}}_{\mathbf{L}} \quad : \quad \lambda \in \mathcal{R} \quad \vee \quad (\lambda \in \mathcal{C} \quad \wedge \quad \Im\mathfrak{m}\lambda < 0)\Big\}. \tag{2.6}$$

The number of elements in the spectrum $\mathcal{S}_{\mathbf{L}}$ is denoted by $\Omega$. An additional benefit of the above definition is that a relatively simple way of ordering and numbering the eigenvalues in the spectrum $\mathcal{S}_{\mathbf{L}}$ can be introduced – as will be shown later.

The roots of the characteristic polynomial $W(\lambda)$ can be multiple – the multiplicity of the $i$-th root $\lambda_i$ is called its algebraic multiplicity and denoted by $n_a(\lambda_i)$, whereby relation $1 \leq n_a(\lambda_i) \leq 2N$ is true.

Analysing the eigenvectors associated with the eigenvalues, let us point out that Eqs. (1.5) are two different eigenproblems (left and right, respectively) with the same eigenvalues and their algebraic multiplicities (identical spectrum), but in general different complex eigenvectors, each of which has dimensions $[N \times 1]$. The corresponding equations resulting from the conjugations of Eqs. (1.5) are also true:

$$\left(\lambda^{*2}\mathbf{M} + \lambda^*\mathbf{C} + \mathbf{K}\right)\mathbf{\Psi}^* = \mathbf{0}, \qquad \left(\lambda^2\mathbf{M}^T + \lambda\mathbf{C}^T + \mathbf{K}^T\right)\mathbf{\Phi}^* = \mathbf{0}. \tag{2.7}$$

A comparison of Eqs. (1.5) and (2.7) shows that if the eigenvectors $\mathbf{\Psi}$ and $\mathbf{\Phi}^*$ are associated with the eigenvalue $\lambda$, then the eigenvectors $\mathbf{\Psi}^*$ and $\mathbf{\Phi}$ are always associated with the conjugate $\lambda^*$; the complex eigenvalue $\lambda$ and its conjugate $\lambda^*$ may correspond to real eigenvectors, then $\mathbf{\Psi} = \mathbf{\Psi}^* \in \mathcal{R}^{N\times 1}$ and $\mathbf{\Phi} = \mathbf{\Phi}^* \in \mathcal{R}^{N\times 1}$ – thus, in this case, a complex pair of eigenvalues corresponds in fact to one right real eigenvector and one left real eigenvector. In contrast, a real eigenvalue always corresponds to real eigenvectors, and it may happen that different real eigenvalues correspond to the same eigenvector. For every eigenvalue, there corresponds at least one left and one right eigenvector; for every left eigenvector, there corresponds a right eigenvector and vice versa. Thus, eigenvectors create separate left and right bases in the eigenspaces.

In the general case, the numbers of eigenvectors resulting from the solution of the eigenproblem (Eqs. (1.5)) may be less than the numbers of eigenvalues together with their multiplicities, i.e., less than $2N$ – in which case we will say that the operator $\mathbf{L}$ is defective, i.e.,

not diagonalizable. The number of linearly independent eigenvectors associated with a given eigenvalue $\lambda_m$ in the spectrum is called its geometric multiplicity $n_g(\lambda_m)$, the number $n_g(\lambda_m)$ is the same for the left and right eigenvectors. So, there is a fundamental relationship:

$$1 \leq n_g(\lambda_m) \leq n_a(\lambda_m) \leq 2N, \qquad m = 1, ..., \Omega. \tag{2.8}$$

So in particular, it may happen that the number of eigenvectors $n_g(\lambda_m)$ associated with a given eigenvalue $\lambda_m$ may be less than the algebraic multiplicity $n_a(\lambda_m)$ of this eigenvalue – if $n_g(\lambda_m) < n_a(\lambda_m)$, then the eigenvalue $\lambda_m$ will be said to be defective; if $n_g(\lambda_m) = n_a(\lambda_m)$, then $\lambda_m$ is non-defective. In general, the relations between algebraic and geometric multiplicities of eigenvalues are shown in Fig. 1 (Leung, 1993).

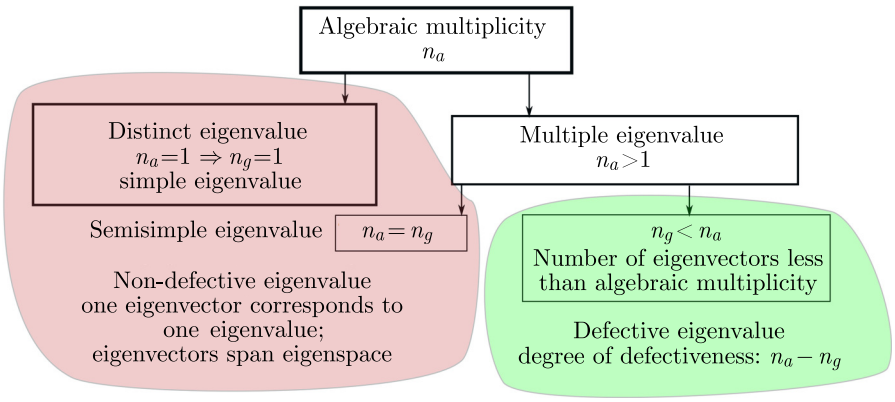


Fig. 1. Relationships between algebraic $n_a$ and geometric $n_g$ multiplicities of eigenvalues and resulting classification (Leung, 1993).

For further analysis, let us write more precisely the eigenproblem corresponding to Eqs. (1.5) with real, non-symmetric matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, which can be represented in two ways. As an eigenproblem with a right eigenvector:

$$\left(\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K}\right)\boldsymbol{\Psi}_i = \mathbf{0}, \qquad i = 1, ..., 2N, \tag{2.9}$$

or with a left eigenvector:

$$\boldsymbol{\Phi}_i^H\left(\lambda_i^2\mathbf{M} + \lambda_i\mathbf{C} + \mathbf{K}\right) = \mathbf{0}, \qquad i = 1, ..., 2N. \tag{2.10}$$

One way to sort the eigenvalues $\lambda_i$ is the order according to non-decreasing real values. The order introduced here is taken from (Krog & Olhoff, 1995; Olhoff *et al.*, 1995; Seyranian *et al.*, 1994):

$$\underbrace{\underbrace{\widetilde{\lambda}_S, ..., \widetilde{\lambda}_{S_0+1}}_{\Im\mathrm{m}\widetilde{\lambda} < 0}\ \overbrace{\widetilde{\lambda}_{S_0} \leq ... \leq \widetilde{\lambda}_1}^{\Im\mathrm{m}\widetilde{\lambda}=\widetilde{\omega}=0}}_{\widetilde{\sigma}_{i+1}\leq\widetilde{\sigma}_i,\ \ \Re\mathrm{e}\widetilde{\lambda}=\widetilde{\sigma}<0}\ \underbrace{\underbrace{\widehat{\lambda}_1 \leq ... \leq \widehat{\lambda}_{U_0}}_{\Im\mathrm{m}\widehat{\lambda}=\widehat{\omega}=0}\ \overbrace{\widehat{\lambda}_{U_0+1}, ..., \widehat{\lambda}_U}^{\Im\mathrm{m}\widehat{\lambda} < 0}}_{\widehat{\sigma}_i\leq\widehat{\sigma}_{i+1},\ \ \Re\mathrm{e}\widehat{\lambda}=\widehat{\sigma}\geq 0}. \tag{2.11}$$

$$\downarrow \widetilde{\lambda}^* \qquad\qquad \downarrow \widehat{\lambda}^*$$

$$\Im\mathrm{m}\widetilde{\lambda}^* > 0 \qquad\qquad \Im\mathrm{m}\widehat{\lambda}^* > 0$$

The specification of the number of solutions is as follows:

$$2N = S_0 + U_0 + 2(S - S_0) + 2(U - U_0). \tag{2.12}$$

In addition, let

$$L = S + U, \tag{2.13}$$

then it can be written:

$$\lambda_i = \begin{cases} \widetilde{\lambda}_{S-i+1} & \text{when } \; i \leq S, \\ \widehat{\lambda}_{i-S} & \text{when } \; S < i \leq L, \end{cases} \qquad i = 1, ..., L. \tag{2.14}$$

In particular, it follows from Eq. (2.14) that: $\lambda_1 = \widetilde{\lambda}_S$, $\lambda_{S+1} = \widehat{\lambda}_1$, $\lambda_L = \widehat{\lambda}_U$.

As mentioned, the essence of the ordering thus introduced in Eq. (2.11) is to sort and number all eigenvalues with non-positive imaginary parts (eigenfrequencies) according to the non-decreasing real parts; in addition, the eigenvalues with negative and non-negative real parts $\mathfrak{Re}\lambda$ have been distinguished – $\widetilde{\sigma}$ for $\mathfrak{Re}\lambda < 0$ and $\widehat{\sigma}$ for $\mathfrak{Re}\lambda \geq 0$, respectively. Complex eigenvalues with positive imaginary parts were ordered analogously to their conjugations. The presented ordering of eigenvalues with non-positive imaginary parts refers to the already introduced notion of a spectrum $\mathcal{S}_{\mathbf{L}}$. The elements of this spectrum are the different roots from among those covered by the two upper brackets in Eq. (2.11).

Suppose that there are $\Omega$ of different eigenvalues ($\Omega \leq L$) in the spectrum of $\mathcal{S}_{\mathbf{L}}$. Taking into account Eq. (2.14), it is convenient to number them as follows:

$$\check{\lambda}_m = \lambda_i, \qquad i = r_m, r_m + 1, r_m + 2, ..., R_m, \quad m = 1, ..., \Omega, \tag{2.15}$$

where $r_1 = 1$, $r_{m+1} = R_m + 1$, $R_\Omega = L$, and $n_a(\check{\lambda}_m) = R_m - r_m + 1$ is algebraic multiplicity of $m$-th eigenvalues. Note that with $r_m = R_m$, $n_a(\check{\lambda}_m) = 1$, i.e., $\check{\lambda}_m = \lambda_i$, $i = r_m$ is a single eigenvalue.

An important issue that is considered in terms of eigenvectors is their scaling formula. In eigenvector sensitivity analysis, the formula also plays an important role. The following scaling formula is used in this study:

$$\boldsymbol{\Psi}_j^T \left(2\lambda_j \mathbf{M} + \mathbf{C}\right) \boldsymbol{\Psi}_j = 1, \qquad \boldsymbol{\Phi}_j^T \left(2\lambda_j^* \mathbf{M}^T + \mathbf{C}^T\right) \boldsymbol{\Phi}_j = 1. \tag{2.16}$$

In the above formulas, the dependence of the vectors, matrices, and eigenvalues on the parameter $\mathbf{h}$ is not shown for clarity of notation. Let us note here, however, that the proposed method of scaling captures a certain relation of the vector with its transpose, leading to unity, i.e., to a scalar, which is no longer a function of the $\mathbf{h}$, and therefore, its derivative is zero. We will use this scaling property written by Eq. (2.16) as an additional condition in determining the derivatives of the eigenvectors – the derivatives of the mentioned formula will combine the scaled eigenvectors and their derivatives.

For any multiple eigenvalue of $\check{\lambda}_m$, Eqs. (2.9) and (2.10) are true, but they are also true for any linear combinations of eigenvectors corresponding to $\check{\lambda}_m$. This fact can be put as follows for a right eigenvector:

$$\check{\boldsymbol{\Psi}}_j^m = \sum_{k=r_m}^{R_m} \theta_{kj}^m \boldsymbol{\Psi}_k^m, \qquad \theta_{kj}^m \in \mathcal{C}, \quad j = r_m, ..., R_m, \quad m = 1, ..., \Omega, \tag{2.17}$$

and for a left eigenvector:

$$\check{\boldsymbol{\Phi}}_j^m = \sum_{k=r_m}^{R_m} \alpha_{kj}^m \boldsymbol{\Phi}_k^m, \qquad \alpha_{kj}^m \in \mathcal{C}, \quad j = r_m, ..., R_m, \quad m = 1, ..., \Omega, \tag{2.18}$$

where the coefficients $\theta_{kj}^m$ and $\alpha_{kj}^m$ form matrices $\boldsymbol{\theta}^m$, $\boldsymbol{\alpha}^m$, respectively. The above two formulas reveal the assumption that the considered multiple eigenvalue $\theta_{kj}^m$ is non-defective, because there exist for them $n_g(\theta_{kj}^m) = n_a(\theta_{kj}^m)$ linearly independent eigenvectors. But, as pointed out earlier, determining the basis of the eigenvectors corresponding to a multiple eigenvalue is not even unambiguous as to direction.

Both Eqs. (2.17) and (2.18) can be written for convenience in the respective matrix forms:

$$\begin{aligned} \check{\boldsymbol{\Psi}}^m &= \boldsymbol{\Psi}^m \boldsymbol{\theta}^m, \qquad \boldsymbol{\theta}^m = \left[\theta_{kj}^m\right], \\ \check{\boldsymbol{\Phi}}^m &= \boldsymbol{\Phi}^m \boldsymbol{\alpha}^m, \qquad \boldsymbol{\alpha}^m = \left[\alpha_{kj}^m\right], \end{aligned} \qquad j,k = r_m, ..., R_m, \quad m = 1, ..., \Omega, \tag{2.19}$$

where the matrices of the complex coefficients $\boldsymbol{\theta}^m$ and $\boldsymbol{\alpha}^m$ are quadratic and have the dimension of the algebraic eigenvalue times $\left[n_a(\check{\lambda}_m) \times n_a(\check{\lambda}_m)\right]$; moreover, these matrices are orthogonal, i.e., $\boldsymbol{\theta}^m = \boldsymbol{\theta}^{mH}$, $\boldsymbol{\alpha}^m = \boldsymbol{\alpha}^{mH}$. In contrast, the other matrices

$$\begin{aligned} \boldsymbol{\Psi}^m &= \left[\boldsymbol{\Psi}_{r_m}, ..., \boldsymbol{\Psi}_{R_m}\right], \qquad \check{\boldsymbol{\Psi}}^m = \left[\check{\boldsymbol{\Psi}}_{r_m}^m, ..., \check{\boldsymbol{\Psi}}_{R_m}^m\right], \\ \boldsymbol{\Phi}^m &= \left[\boldsymbol{\Phi}_{r_m}, ..., \boldsymbol{\Phi}_{R_m}\right], \qquad \check{\boldsymbol{\Phi}}^m = \left[\check{\boldsymbol{\Phi}}_{r_m}^m, ..., \check{\boldsymbol{\Phi}}_{R_m}^m\right], \end{aligned} \qquad m = 1, ..., \Omega, \tag{2.20}$$

have dimensions $\left[N \times n_a(\check{\lambda}_m)\right]$. The columns of the matrix $\boldsymbol{\Psi}^m$ are right eigenvectors, and the columns of the matrix $\boldsymbol{\Phi}^m$ are left eigenvectors – these correspond to the multiple eigenvalue $\check{\lambda}_m$ and are the result of solving the eigenproblems (Eqs. (2.9) and (2.10)).

It follows from the assumption that $\check{\lambda}_m$ is a non-defective eigenvalue that the coefficient matrices of $\boldsymbol{\theta}^m$ and $\boldsymbol{\alpha}^m$ are of full rank, i.e:

$$\operatorname{rank}\left(\boldsymbol{\theta}^m\right) = \operatorname{rank}\left(\boldsymbol{\alpha}^m\right) = n_a\left(\check{\lambda}_m\right), \qquad m = 1, ..., \Omega. \tag{2.21}$$

In the remaining part of the text, it will be assumed that the right eigenvectors $\check{\boldsymbol{\Psi}}_j^m$ of Eq. (2.17) and the left eigenvectors $\check{\boldsymbol{\Phi}}_j^m$ of Eq. (2.18), calculated as linear combinations of the corresponding eigenvectors $\boldsymbol{\Psi}_j^m$, $\boldsymbol{\Phi}_j^m$ obtained directly from Eqs. (2.9) and (2.10), were scaled according to Eq. (2.16).

The following equations also remain true for the adopted ordering and designations:

$$\left(\check{\lambda}_m^2 \mathbf{M} + \check{\lambda}_m \mathbf{C} + \mathbf{K}\right) \check{\boldsymbol{\Psi}}_j^m = \mathbf{0}, \qquad j = r_m, ..., R_m, \quad m = 1, ..., \Omega, \tag{2.22}$$

$$\left(\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}\right) \check{\boldsymbol{\Psi}}_j^m = \mathbf{0}, \qquad j = r_m, ..., R_m, \quad m = 1, ..., \Omega, \tag{2.23}$$

$$\check{\boldsymbol{\Phi}}_s^{mH} \left(\check{\lambda}_m^2 \mathbf{M} + \check{\lambda}_m \mathbf{C} + \mathbf{K}\right) = \mathbf{0}, \qquad s = r_m, ..., R_m, \quad m = 1, ..., \Omega, \tag{2.24}$$

$$\check{\boldsymbol{\Phi}}_s^{mH} \left(\lambda_s^2 \mathbf{M} + \lambda_s \mathbf{C} + \mathbf{K}\right) = \mathbf{0}, \qquad s = r_m, ..., R_m, \quad m = 1, ..., \Omega. \tag{2.25}$$

## 3. Directional derivatives of eigenvalues

To compute the directional derivative in the Gâteaux sense, one has to perturb a mapping along the $\mathbf{e}$ direction. In order to find the value of the function at the point $\mathbf{h} + \epsilon\mathbf{e}$, the Taylor expansion around the point $\mathbf{h}$ restricted to the linear part with respect to $\epsilon$ is used. Thus, for $m = 1, ..., \Omega$ and $j = r_m, ..., R_m$ the following expansions exists:

$$\lambda_j\left(\mathbf{h} + \epsilon\mathbf{e}\right) = \check{\lambda}_m\left(\mathbf{h}\right) + \epsilon\mu_j, \qquad \lambda_j^*\left(\mathbf{h} + \epsilon\mathbf{e}\right) = \check{\lambda}_m^*\left(\mathbf{h}\right) + \epsilon\mu_j^*, \tag{3.1}$$

$$\check{\boldsymbol{\Psi}}_j^m\left(\mathbf{h} + \epsilon\mathbf{e}\right) = \check{\boldsymbol{\Psi}}_j^m\left(\mathbf{h}\right) + \epsilon\boldsymbol{\Gamma}_j, \qquad \check{\boldsymbol{\Phi}}_j^m\left(\mathbf{h} + \epsilon\mathbf{e}\right) = \check{\boldsymbol{\Phi}}_j^m\left(\mathbf{h}\right) + \epsilon\boldsymbol{\Pi}_j, \tag{3.2}$$

where $\mu_j \in \mathcal{C}$, $\boldsymbol{\Gamma}_j = [\Gamma_{rj}]$ and $\Gamma_{rj} \in \mathcal{C}$, $\boldsymbol{\Pi}_j = [\Pi_{rj}]$ and $\Pi_{rj} \in \mathcal{C}$, $r = 1, ..., N$ are the sought directional derivatives of the $j$-th eigenvalue and the associated right and left directional derivatives of the eigenvector, respectively; it follows from the above notations that the derivatives mentioned here are complex, so they consist of derivatives of real parts and derivatives of imaginary parts.

The basis for the derivation of directional derivatives in the Gâteaux sense of the eigenvalues of Eq. (2.23) and (2.22) is to write them at the point $\mathbf{h} + \epsilon\mathbf{e}$ for $j = r_m, ..., R_m$, $m = 1, ..., \Omega$:

$$\left[\check{\lambda}_m^2(\mathbf{h} + \epsilon\mathbf{e})\mathbf{M}(\mathbf{h} + \epsilon\mathbf{e}) + \check{\lambda}_m(\mathbf{h} + \epsilon\mathbf{e})\mathbf{C}(\mathbf{h} + \epsilon\mathbf{e}) + \mathbf{K}(\mathbf{h} + \epsilon\mathbf{e})\right] \check{\boldsymbol{\Psi}}_j^m(\mathbf{h} + \epsilon\mathbf{e}) = \mathbf{0}. \tag{3.3}$$

By substituting Eqs. (3.1), (3.2), (1.4) into Eq. (3.3) one obtains for $j = r_m, ..., R_m$, $m = 1, ..., \Omega$:

$$\left[ \left( \check{\lambda}_m^2 + 2\epsilon \check{\lambda}_m \mu_j + \epsilon^2 \mu_j^2 \right) \left( \mathbf{M} + \epsilon \sum_{p=1}^{N_p} \frac{\partial \mathbf{M}}{\partial h_p} e_p \right) \right.$$
$$\left. + \left( \check{\lambda}_m + \epsilon \mu_j \right) \left( \mathbf{C} + + \epsilon \sum_{p=1}^{N_p} \frac{\partial \mathbf{C}}{\partial h_p} e_p \right) + \mathbf{K} + \epsilon \sum_{p=1}^{N_p} \frac{\partial \mathbf{K}}{\partial h_p} e_p \right] \left( \check{\mathbf{\Psi}}_j^m + \epsilon \mathbf{\Gamma}_j \right) = \mathbf{0}. \tag{3.4}$$

Further transformations of Eq. (3.4) take into account that it is true for any $\epsilon > 0$, non-linear terms relative to $\epsilon$ are omitted, and that Eq. (3.2) is true, and also Eq. (2.23) is substituted. After simplifying, the sums give the following equations:

$$\left( \check{\lambda}_m^2 \mathbf{M} + \check{\lambda}_m \mathbf{C} + \mathbf{K} \right) \mathbf{\Gamma}_j + \left( \check{\lambda}_m^2 \sum_{p=1}^{N_p} \frac{\partial \mathbf{M}}{\partial h_p} e_p + \check{\lambda}_m \sum_{p=1}^{N_p} \frac{\partial \mathbf{C}}{\partial h_p} e_p + \sum_{p=1}^{N_p} \frac{\partial \mathbf{K}}{\partial h_p} e_p \right) \check{\mathbf{\Psi}}_j^m$$
$$+ \mu_j \left( 2\check{\lambda}_m \mathbf{M} + \mathbf{C} \right) \check{\mathbf{\Psi}}_j^m = \mathbf{0}, \qquad j = r_m, ..., R_m, \quad m = 1, ..., \Omega. \tag{3.5}$$

Before discussing further, let us introduce for convenience the following designations for $m = 1, ..., \Omega$:

$$\widehat{\mathbf{E}}^m = \check{\lambda}_m^2 \mathbf{M} + \check{\lambda}_m \mathbf{C} + \mathbf{K}, \qquad \widehat{\mathbf{G}}^m = 2\check{\lambda}_m \mathbf{M} + \mathbf{C},$$
$$\widehat{\mathbf{F}}^m = \sum_{p=1}^{N_p} \left( \check{\lambda}_m^2 \frac{\partial \mathbf{M}}{\partial h_p} + \check{\lambda}_m \frac{\partial \mathbf{C}}{\partial h_p} + \frac{\partial \mathbf{K}}{\partial h_p} \right) e_p. \tag{3.6}$$

Taking into account the above, Eq. (3.5) can be written in the form:

$$\widehat{\mathbf{E}}^m \mathbf{\Gamma}_j + \widehat{\mathbf{F}}^m \check{\mathbf{\Psi}}_j^m + \widehat{\mathbf{G}}^m \check{\mathbf{\Psi}}_j^m \mu_j = \mathbf{0}, \qquad j = r_m, ..., R_m, \quad m = 1, ..., \Omega. \tag{3.7}$$

Multiplying Eq. (3.7) left-hand by $\mathbf{\Phi}_s^H$, we notice that the first component of $\mathbf{\Phi}_s^H \widehat{\mathbf{E}}^m$ disappears, because there is Eq. (2.24), and Eq. (2.17) should also be considered here. Introducing the simplified notations $\mathbf{A}^m = [a_{sk}^m]$, $\mathbf{B}^m = [b_{sk}^m]$ where

$$a_{sk}^m = \mathbf{\Phi}_s^H \widehat{\mathbf{F}}^m \mathbf{\Psi}_k, \qquad b_{sk}^m = \mathbf{\Phi}_s^H \widehat{\mathbf{G}}^m \mathbf{\Psi}_k, \tag{3.8}$$

finally, Eq. (3.5) is obtained in the form:

$$\sum_{k=r_m}^{R_m} \left( a_{sk}^m + \mu_j b_{sk}^m \right) \theta_{kj}^m = 0, \qquad j, s = r_m, ..., R_m, \quad m = 1, ..., \Omega. \tag{3.9}$$

The following additional eigenproblem arises from Eq. (3.9):

$$\left( \mathbf{A}^m + \mu_j \mathbf{B}^m \right) \boldsymbol{\theta}_j^m = \mathbf{0}, \qquad j = r_m, ..., R_m, \quad m = 1, ..., \Omega. \tag{3.10}$$

The matrices $\mathbf{A}^m$ and $\mathbf{B}^m$ are, in general, asymmetric and complex, and $\boldsymbol{\theta}_j^m = \left[ \theta_{kj}^m \right]$. From Eq. (3.10), one computes $n_a(\check{\lambda}_m)$ of the eigenvalues, i.e., derivatives of $\mu_{r_m}, ..., \mu_{R_m}$, which are being sought. An important observation is that if the matrices $\mathbf{A}^m$ and $\mathbf{B}^m$ are both diagonal (off-diagonal terms are equal to zero), the eigenvalues $\mu_j$, i.e., Gatêaux directional derivatives of multiple eigenvalues $\check{\lambda}_m$ are the same as the traditional Fréchet derivatives. The diagonal form

of matrices $\mathbf{A}^m$ and $\mathbf{B}^m$ occurs when matrices $\mathbf{F}^m$ and $\mathbf{G}^m$ from Eq. (3.6) are simultaneously diagonalizable by transformations (Eq. (3.8)). Of course, for a single eigenvalue $n_a(\check{\lambda}_m) = 1$ (scalar equation), these derivatives are also the same.

Equation (3.10) has, in practice, a low dimension associated with the algebraic multiplicities of the multiple eigenvalue $\check{\lambda}_m$. However, for a single (simple) eigenvalue, when, for a given $m$, the algebraic multiplicity $n_a(\check{\lambda}_m) = 1$ (then $j = k = s = r_m$), Eq. (3.10) turns into a scalar equation from which the directional derivative $\mu_j$ of the simple eigenvalue $\lambda_j$ is calculated. Thus, the derived relations leading to (Eq. (3.10)) constitute an algorithm for calculating the directional derivatives of the eigenvalues of the eigenproblem (Eq. (2.22)) regardless of their multiplicity.

## 4. Directional derivatives of eigenvectors

The derivation of the derivative of the right eigenvector is shown here – this may be done analogously for the left eigenvector. The source is Eq. (3.7), because in it the derivative of the eigenvector $\mathbf{\Gamma}_j$ appears in the process of differentiation. Let us rewrite Eq. (3.7) in a more convenient form:

$$\widehat{\mathbf{E}}^m\mathbf{\Gamma}_j = -\widehat{\mathbf{F}}^m\check{\mathbf{\Psi}}_j^m - \widehat{\mathbf{G}}^m\check{\mathbf{\Psi}}_j^m\mu_j, \qquad j = r_m, ..., R_m, \quad m = 1, ..., \Omega. \tag{4.1}$$

If the derivatives of the eigenvalues $\mu_j$ are already known, as they are calculated from Eq. (3.10), the right-hand side of Eq. (4.1) is different from zero and it is theoretically possible to determine $\mathbf{\Gamma}_j$, i.e., the derivatives of the right eigenvector.

However, the fundamental difficulty in Eq. (4.1) relates to the fact that, since $\check{\lambda}_m$ is an eigenvalue with multiplicity $n_a(\check{\lambda}_m)$, so the matrix $\mathbf{E}^m$ is singular (rank $\mathbf{E}^m$ is reduced by $n_a(\check{\lambda}_m)$) – it is therefore not possible to simply compute $\mathbf{\Gamma}_j$. Overcoming this difficulty has been the subject of numerous research papers and related computational algorithms of varying complexity and difficulty of application, both in configuration space and state space. An extensive review of the papers in this area is given in (Choi *et al.*, 2004).

The algorithms presented in (Kim *et al.*, 1999a; 1999b; Choi *et al.*, 2004; Lee *et al.*, 1999a; 1999b) are used in the paper but extended to non-symmetric matrices in QEP. The idea is to use the eigenvector scaling Eq. (2.16) for the right eigenvector as a constraint condition – because, as mentioned, this equation, when differentiated, ties the vector $\mathbf{\Psi}$ itself to its derivative $\mathbf{\Gamma}_j$. The scaling condition is written here in the formula for multiple eigenvalues:

$$\check{\mathbf{\Psi}}_j^{mT}(\mathbf{h})\left[2\check{\lambda}_m(\mathbf{h})\mathbf{M}(\mathbf{h}) + \mathbf{C}(\mathbf{h})\right]\check{\mathbf{\Psi}}_j^m(\mathbf{h}) = 1, \qquad j = r_m, ..., R_m, \quad m = 1, ..., \Omega. \tag{4.2}$$

Thus, let us expand the scaling condition (Eq. (4.2)) at the point $\mathbf{h}$ and in the direction $\mathbf{e}$ (analogous to Eq. (3.3)), however, under the assumption that the vectors $\check{\mathbf{\Psi}}_j^m$ are already scaled according to Eq. (4.2), (the scaling should therefore be done after solving Eq. (3.10) and performing a linear combination defined in Eq. (2.19)), so for $j = r_m, ..., R_m,\ m = 1, ..., \Omega$:

$$\check{\mathbf{\Psi}}_j^{mT}(\mathbf{h}+\epsilon\mathbf{e})\left[2\check{\lambda}_m(\mathbf{h}+\epsilon\mathbf{e})\mathbf{M}(\mathbf{h}+\epsilon\mathbf{e}) + \mathbf{C}(\mathbf{h}+\epsilon\mathbf{e})\right]\check{\mathbf{\Psi}}_j^m(\mathbf{h}+\epsilon\mathbf{e}) = 1. \tag{4.3}$$

Substituting Eqs. (3.1), (3.2), (1.4) to Eq. (4.3), the following is obtained:

$$\left(\check{\mathbf{\Psi}}_j^{mT} + \epsilon\mathbf{\Gamma}_j^T\right)\left[2\left(\check{\lambda}_m + \epsilon\mu_j\right)\left(\mathbf{M} + \epsilon\sum_{p=1}^{N_p}\frac{\partial\mathbf{M}}{\partial h_p}e_p\right) + \mathbf{C} + \epsilon\sum_{p=1}^{N_p}\frac{\partial\mathbf{C}}{\partial h_p}e_p\right]\left(\check{\mathbf{\Psi}}_j^m + \epsilon\mathbf{\Gamma}_j\right) = 1. \tag{4.4}$$

Further transformations of Eq. (4.4) take into account that it is true for any $\epsilon > 0$, nonlinear members with respect to $\epsilon$ are omitted, and that Eq. (3.2) is considered, also Eqs. (2.23), (2.16) are substituted. The final expression after ordering the sums is obtained in the form:

$$\breve{\mathbf{\Psi}}_j^{mT}\left[2\breve{\lambda}_m\left(\mathbf{M}+\mathbf{M}^T\right)+\mathbf{C}+\mathbf{C}^T\right]\mathbf{\Gamma}_j+2\breve{\mathbf{\Psi}}_j^{mT}\mathbf{M}\breve{\mathbf{\Psi}}_j^m\mu_j$$
$$=-\breve{\mathbf{\Psi}}_j^{mT}\left(2\breve{\lambda}_m\sum_{p=1}^{N_p}\frac{\partial\mathbf{M}}{\partial h_p}e_p+\sum_{p=1}^{N_p}\frac{\partial\mathbf{C}}{\partial h_p}e_p\right)\breve{\mathbf{\Psi}}_j^m. \tag{4.5}$$

For clarity let us introduce the notations:

$$\widehat{\mathbf{H}}^m=\widehat{\mathbf{G}}^{mT}+\widehat{\mathbf{G}}^m,\qquad \widehat{\mathbf{X}}^m=\sum_{p=1}^{N_p}\left(2\breve{\lambda}_m\frac{\partial\mathbf{M}}{\partial h_p}+\frac{\partial\mathbf{C}}{\partial h_p}\right)e_p,\quad m=1,...,\Omega. \tag{4.6}$$

Finally, Eq. (4.5) takes a simpler form:

$$\breve{\mathbf{\Psi}}_j^{mT}\widehat{\mathbf{H}}^m\mathbf{\Gamma}_j+2\breve{\mathbf{\Psi}}_j^{mT}\mathbf{M}\breve{\mathbf{\Psi}}_j^m\mu_j=-\breve{\mathbf{\Psi}}_j^{mT}\widehat{\mathbf{X}}^m\breve{\mathbf{\Psi}}_j^m,\qquad j=r_m,...,R_m,\quad m=1,...,\Omega. \tag{4.7}$$

Note that Eqs. (4.1) and (4.7) tie the derivatives of the eigenvalue of $\mu_j$ and the derivatives of the eigenvectors of $\mathbf{\Gamma}_j$ – so let us write the two equations together, finally for $j=r_m,...,R_m$, $m=1,...,\Omega$ we have

$$\begin{cases}\widehat{\mathbf{E}}^m\mathbf{\Gamma}_j+\widehat{\mathbf{G}}^m\breve{\mathbf{\Psi}}_j^m\mu_j=-\widehat{\mathbf{F}}^m\breve{\mathbf{\Psi}}_j^m,\\ \breve{\mathbf{\Psi}}_j^{mT}\widehat{\mathbf{H}}^m\mathbf{\Gamma}_j+2\breve{\mathbf{\Psi}}_j^{mT}\mathbf{M}\breve{\mathbf{\Psi}}_j^m\mu_j=-\breve{\mathbf{\Psi}}_j^{mT}\widehat{\mathbf{X}}^m\breve{\mathbf{\Psi}}_j^m.\end{cases} \tag{4.8}$$

The above system of equations is still unsolvable for a multiple eigenvalue, i.e., with $n_a\left(\breve{\lambda}_m\right)>1$ – the second equation is a scalar equation and, as noted earlier, the rank of the matrix $\widehat{\mathbf{E}}^m$ is equal:

$$\operatorname{rank}\left(\widehat{\mathbf{E}}^m\right)=N-n_a\left(\breve{\lambda}_m\right).$$

Let us introduce the following diagonal auxiliary matrices:

$$\boldsymbol{\mu}^m=\begin{bmatrix}\mu_{r_m}&0&\cdots&0\\0&\mu_{r_m+1}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\mu_{R_m}\end{bmatrix},\qquad \mathbf{Z}^m=2\begin{bmatrix}\widehat{Z}_{r_m}^m&0&\cdots&0\\0&\widehat{Z}_{r_m+1}^m&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\widehat{Z}_{R_m}^m\end{bmatrix},$$
$$\mathbf{V}^m=\begin{bmatrix}\widehat{V}_{r_m}^m&0&\cdots&0\\0&\widehat{V}_{r_m+1}^m&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\widehat{V}_{R_m}^m\end{bmatrix},\qquad \widehat{Z}_j^m=\breve{\mathbf{\Psi}}_j^{mT}\mathbf{M}\breve{\mathbf{\Psi}}_j^m,\qquad \widehat{V}_j^m=\breve{\mathbf{\Psi}}_j^{mT}\widehat{\mathbf{X}}^m\breve{\mathbf{\Psi}}_j^m, \tag{4.9}$$

and the derivative matrix:

$$\mathbf{\Gamma}^m=\left[\mathbf{\Gamma}_{r_m}\ \ \mathbf{\Gamma}_{r_m+1}\ \ \cdots\ \ \mathbf{\Gamma}_{R_m}\right]. \tag{4.10}$$

The dimensions of the predefined matrices are as follows:

$$\widehat{\mathbf{E}}^m\longrightarrow[N\times N],\qquad \widehat{\mathbf{G}}^m\breve{\mathbf{\Psi}}^m\longrightarrow\left[N\times n_a\left(\breve{\lambda}_m\right)\right],$$
$$\breve{\mathbf{\Psi}}^{mT}\widehat{\mathbf{H}}^m\longrightarrow\left[n_a\left(\breve{\lambda}_m\right)\times N\right],\qquad \mathbf{Z}^m\longrightarrow\left[n_a\left(\breve{\lambda}_m\right)\times n_a\left(\breve{\lambda}_m\right)\right],$$
$$\mathbf{\Gamma}^m\longrightarrow\left[N\times n_a\left(\breve{\lambda}_m\right)\right],\qquad \boldsymbol{\mu}^m\longrightarrow\left[n_a\left(\breve{\lambda}_m\right)\times n_a\left(\breve{\lambda}_m\right)\right],$$
$$\widehat{\mathbf{F}}^m\breve{\mathbf{\Psi}}^m\longrightarrow\left[N\times n_a\left(\breve{\lambda}_m\right)\right],\qquad \mathbf{V}^m\longrightarrow\left[n_a\left(\breve{\lambda}_m\right)\times n_a\left(\breve{\lambda}_m\right)\right].$$

Note that the rank of the individual submatrices in Eq. (4.11) are as follows:

$$\operatorname{rank}\left(\check{\boldsymbol{\Psi}}^{m}\right)=\operatorname{rank}\left(\check{\boldsymbol{\Psi}}^{mT}\right)=n_a\left(\check{\lambda}_m\right),$$

$$\operatorname{rank}\left(\mathbf{Z}^{m}\right)=\operatorname{rank}\left(\widehat{\mathbf{G}}^{m}\check{\boldsymbol{\Psi}}^{m}\right)=\operatorname{rank}\left(\check{\boldsymbol{\Psi}}^{mT}\widehat{\mathbf{H}}^{m}\right)=n_a\left(\check{\lambda}_m\right).$$

Using these definitions, the following matrix equation can be written in the form:

$$\underbrace{\begin{bmatrix}\widehat{\mathbf{E}}^{m} & \widehat{\mathbf{G}}^{m}\check{\boldsymbol{\Psi}}^{m}\\ \check{\boldsymbol{\Psi}}^{mT}\widehat{\mathbf{H}}^{m} & \mathbf{Z}^{m}\end{bmatrix}}_{\mathbf{S}^{m}}\underbrace{\begin{bmatrix}\boldsymbol{\Gamma}^{m}\\ \boldsymbol{\mu}^{m}\end{bmatrix}}_{\mathbf{Y}^{m}}=\underbrace{-\begin{bmatrix}\widehat{\mathbf{F}}^{m}\check{\boldsymbol{\Psi}}^{m}\\ \mathbf{V}^{m}\end{bmatrix}}_{\mathbf{R}^{m}}. \tag{4.11}$$

Taking into account the previously listed dimensions of the component matrices in Eq. (4.11), the following relations occur:

$$\mathbf{S}^{m}\longrightarrow\left[\left(N+n_a\left(\check{\lambda}_m\right)\right)\times\left(N+n_a\left(\check{\lambda}_m\right)\right)\right],\qquad\mathbf{Y}^{m},\mathbf{R}^{m}\longrightarrow\left[\left(N+n_a\left(\check{\lambda}_m\right)\right)\times n_a\left(\check{\lambda}_m\right)\right].$$

The key observation here is that the quadratic and complex coefficient matrix $\mathbf{S}^m$ is not singular (the method of proof is shown in (Kim *et al.*, 1999b)), i.e., it is invertible and thus, there is an unambiguous solution to the system (Eq. (4.11)) – in this equation, the unknown is the matrix $\mathbf{Y}^m$; moreover, the system of algebraic Eq. (4.11) is well-conditioned. The algorithm presented here leads to the simultaneous determination of the directional derivatives of the right eigenvectors $\boldsymbol{\Gamma}_j$ and the redetermination of the directional derivative $\mu_j$ associated with the eigenvalue of $\lambda_j$ by solving a linear algebraic non-symmetric system of complex Eq. (4.11). For a single eigenvalue, when for a given $m$ the algebraic multiplicity $n_a(\check{\lambda}_m)=1$, the system of Eq. (4.11) also allows us to calculate the desired quantities. Thus, the derived relations constitute an algorithm for computing the directional derivatives of the eigenvectors of Eq. (2.22), irrespective of the eigenvalue multiplicity.

The structure of Eq. (4.11) also indicates that the vector of the directional derivative of the eigenvector $\boldsymbol{\Gamma}_j$ is determined for the same multiplier as the corresponding eigenvector of $\check{\boldsymbol{\Psi}}_j^m$.

## 5. Additional requirements

It is important to pick up the significant practical problems associated with the determination and recognition of multiple eigenvalues of $\lambda_m$ (see, e.g., (Seyranian *et al.*, 1994)). Numerical procedures iterate over the eigenvalues of $\lambda_i$ and the corresponding eigenvectors $\boldsymbol{\Psi}_i$, $\boldsymbol{\Phi}_i$ with some accuracy. Most often, only the tolerance of the $\epsilon_\lambda^{\mathrm{tol}}=1\times10^{-5}\div1\times10^{-3}$ eigenvalue determination is established. Structural models, created at the design stage and characterized by symmetries, may exhibit the presence of multiple eigenvalues, or there may be very closely lying single eigenvalues. This raises the problem of identifying multiple eigenvalues in the spectrum. Thus, an additional algorithm for the recognition of multiple eigenvalues after ordering and renumbering the eigenvalues according to Eq. (2.11) must be applied to the derivation formulas presented.

The simplest way to do this is to take the number $\epsilon_m^{\mathrm{tol}}$ such that $\epsilon_\lambda^{\mathrm{tol}}<\epsilon_m^{\mathrm{tol}}\ll1$ as the tolerance for recognising multiple eigenvalues. Thus, referring to the numbering of Eq. (2.14) we will say that two eigenvalues are multiple when:

$$|\lambda_i-\lambda_{i+1}|\le\epsilon_m^{\mathrm{tol}}, \tag{5.1}$$

which implies that they are at a distance less than $\epsilon_m^{\mathrm{tol}}$ on the complex plane $\sigma-\omega$. Another proposal is to adopt separate distance conditions for eigendamping and eigenfrequency:

$$|\mathfrak{Re}\lambda_i-\mathfrak{Re}\lambda_{i+1}|\le\epsilon_m^{\mathrm{tol}_\sigma}\qquad\text{and}\qquad|\mathfrak{Im}\lambda_i-\mathfrak{Im}\lambda_{i+1}|\le\epsilon_m^{\mathrm{tol}_\omega}, \tag{5.2}$$

which means that two eigenvalues are multiple if they lie in the rectangle $\epsilon_m^{\mathrm{tol}_\sigma} \times \epsilon_m^{\mathrm{tol}_\omega}$ on the complex plane $\sigma - \omega$.

It should be noted that, in general cases, the definitions given above do not allow explicit grouping of multiple eigenvalues, but they do allow such grouping in many analyses of typical structures with clearly separated multiple eigenvalues in the spectrum. Unfortunately, the values of tolerances $\epsilon_m^{\mathrm{tol}}$, $\epsilon_m^{\mathrm{tol}_\sigma}$, $\epsilon_m^{\mathrm{tol}_\omega}$ may be problem dependent.

## 6. Numerical examples

### 6.1. Introduction

As an example of a system with multiple eigenvalues, we will present the planar frame shown in Fig. 2. The geometric dimensions are as follows: beam span $L = 16.16855\,\mathrm{m}$, height of columns $H = 5\,\mathrm{m}$. All cross-sections are square in shape. The following parameters were distinguished for further analysis: width of cross-sections $b = 0.4\,\mathrm{m}$, height of column cross-sections $h_s = 0.4\,\mathrm{m}$, height of beam cross-sections $h_r = 0.4\,\mathrm{m}$. In addition, two local zones were distinguished: 1) the right frame node – the cross-section height of these elements $h_n = 0.4\,\mathrm{m}$, and 2) the right column fixed – the cross-section height $h_f = 0.4\,\mathrm{m}$. All mentioned zones span over along one finite beam element.

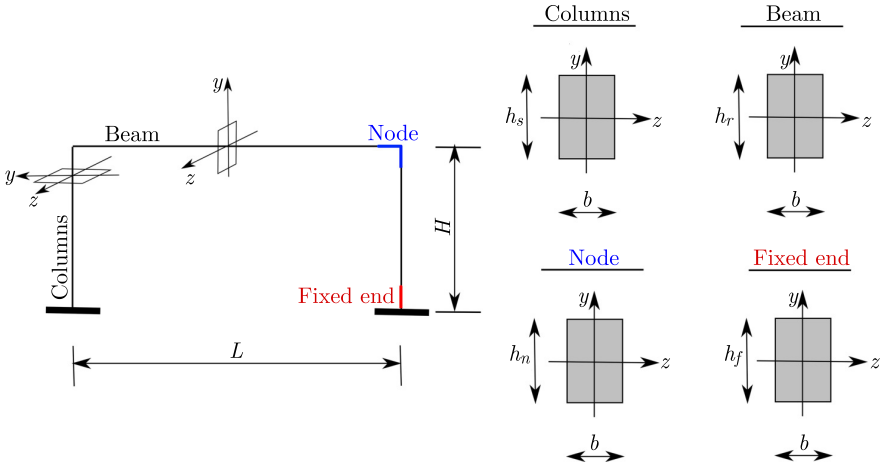


Fig. 2. Flat symmetric frame with fixed columns – geometric and cross-section dimensions.

It was assumed that the frame was made of reinforced concrete: Young's modulus of the material equal to $E = 30\,\mathrm{GPa}$, Poisson's ratio $\nu = 0.2$ and a density $\rho = 2400\,\mathrm{kg/m^3}$. In addition, for simplicity, a damping matrix according to Rayleigh was assumed, i.e., $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$, where $\alpha = 2 \times 10^{-3}$, $\beta = 5.2 \times 10^{-4}$.

The geometric dimensions and cross-sections are set so that the first eigenvalue of $\lambda_1$ corresponding to the antisymmetric mode shape is equal to $\lambda_2$ corresponding to the symmetric mode shape, i.e., $\lambda_1 = \lambda_2$ – Fig. 3. Finally, the symmetric mode shapes correspond to $\lambda_2$ and $\lambda_4$, while the antisymmetric mode shapes correspond to $\lambda_1$ and $\lambda_3$.

In the analysed frame, the matrices of the system $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are symmetric and the eigenmodes are real (classical) – which makes them easier to visualize and in this case the right and left eigenvectors are equal $\mathbf{\Phi} = \mathbf{\Psi} \in \mathcal{R}^N$. The assumptions made do not change the fact that the multiple eigenvalue $\lambda_1 = \lambda_2$ (i.e., $n_a(\lambda_1) = 2$) is differentiable in the Gâteaux sense but is not always differentiable in the Fréchet sense. The tolerance for recognizing multiple eigenvalues in Eq. (5.1) is set to $\epsilon_m^{\mathrm{tol}} = 1 \times 10^{-5}$.

To solve the derivatives shown, a programme in MATLAB was developed to calculate the directional derivatives of the eigenvalues and eigenvectors. The procedure for calculating eigenvalues and eigenvectors for QEP shown in (Hammarling *et al.*, 2013) was used.

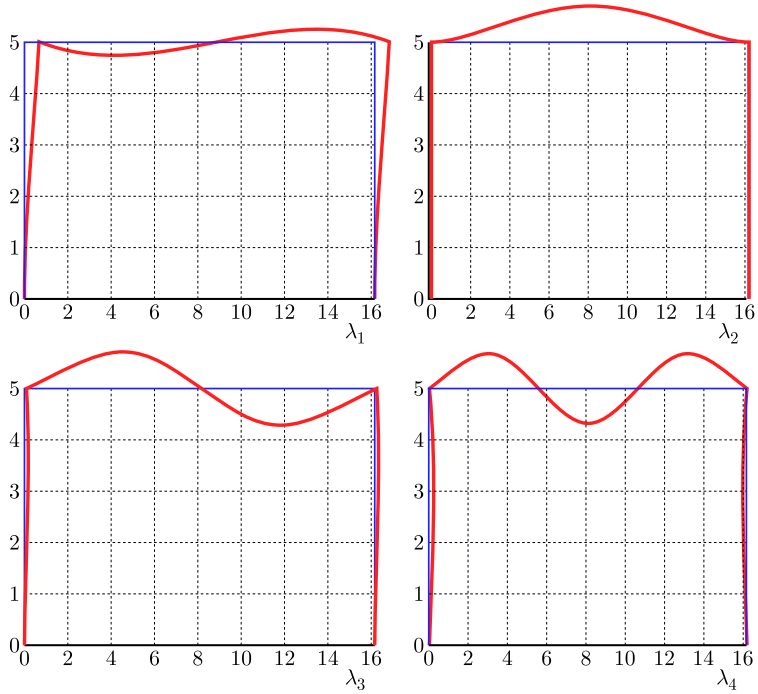


Fig. 3. Eigenmode shapes for consecutive eigenvalues.

### 6.2. Numerical example 1: symmetric changes of the frame

Let the vector of parameters against which the derivatives are calculated be equal:

$$\mathbf{h} = [L, H, b, h_s, h_r], \qquad \mathbf{h} \in \mathcal{R}^5. \tag{6.1}$$

Thus, the dimension of parameter space is equal to $N_p = 5$. Table 1 shows directional derivatives with respect to canonical basis vectors (for example, the basis vector $\mathbf{e}_3 = [0, 0, 1, 0, 0]$). It is noteworthy that a change in any component of the vector $\mathbf{h}$ results in a symmetric change in the geometry and cross sections of the frame members. In such a case, the Gâteaux directional derivatives are the same as the Fréchet derivatives even for multiple eigenvalues, i.e.:

$$\frac{\partial \lambda_i}{\partial h_p} \equiv \mu_i(\mathbf{h}, \mathbf{e}_p), \qquad i = 1, ..., 4, \qquad p = 1, ..., 5.$$

The reason for this property is that the matrices $\mathbf{A}^m$ and $\mathbf{B}^m$ are both diagonal in Eq. (3.10). The diagonal form of matrices $\mathbf{A}^m$ and $\mathbf{B}^m$ occurs because matrices $\mathbf{F}^m$ and $\mathbf{G}^m$ from Eq. (3.6) are

Table 1. Eigenvalues and their directional derivatives along the vectors of canonical basis $\mathbf{e}_p$ of the parameter space $\mathbf{h} = [L, H, b, h_s, h_r]$.

| $\lambda_i$ | Deflected shape | $\sigma_i + \omega_i$ | $\frac{\partial \lambda_i}{\partial L} \equiv \mu_i(\mathbf{h}, \mathbf{e}_1)$ | $\frac{\partial \lambda_i}{\partial H} \equiv \mu_i(\mathbf{h}, \mathbf{e}_2)$ | $\frac{\partial \lambda_i}{\partial b} \equiv \mu_i(\mathbf{h}, \mathbf{e}_3)$ | $\frac{\partial \lambda_i}{\partial h_s} \equiv \mu_i(\mathbf{h}, \mathbf{e}_4)$ | $\frac{\partial \lambda_i}{\partial h_r} \equiv \mu_i(\mathbf{h}, \mathbf{e}_5)$ |
|---|---|---|---|---|---|---|---|
| $\lambda_1$ | antisym frame | −0.20516 +28.0212i | +0.01665 −1.14285i | +0.10935 −7.50359i | −1.01922 +69.9413i | −0.26076 +17.8940i | +0.26076 −17.8940i |
| $\lambda_2$ | sym beam | −0.20516 +28.0212i | +0.04644 −3.18711i | +0.01275 −0.87524i | −1.01598 +69.7185i | −0.15744 +10.8039i | +0.15744 −10.8039i |
| $\lambda_3$ | antisym beam | −1.69186 +80.6252i | +0.37242 −8.87510i | +0.14323 −3.41343i | −8.38976 +199.937i | −0.97417 +23.2155i | +0.97417 −23.2155i |
| $\lambda_4$ | sym beam | −6.37071 +156.392i | +1.41240 −17.3101i | +0.49267 −6.03818i | −31.4011 +384.846i | −2.37618 +29.1220i | +2.37618 −29.1220i |

simultaneously diagonalizable by transformations (Eq. (3.8)), and to explain it deeply, this is because Fréchet derivatives of system matrices ($\partial\mathbf{M}/\partial h_p$, $\partial\mathbf{C}/\partial h_p$, $\partial\mathbf{K}/\partial h_p$ – see Eq. (3.6)) are simultaneously diagonalizable.

### 6.3. Numerical example 2: asymmetrical changes of the frame

In contrast to the previous example, let the vector of parameters against which the derivatives are calculated be equal:

$$\mathbf{h} = [h_n, h_f], \qquad \mathbf{h} \in \mathcal{R}^2. \tag{6.2}$$

Thus, the dimension of parameter space is equal to $N_p = 2$. The canonical basis vectors are equal to $\mathbf{e}_1 = [1, 0]$ and $\mathbf{e}_2 = [0, 1]$. It is worth noting that, in this example, a change in any component of the vector $\mathbf{h}$ results in an asymmetric change in the height of the cross-sections of the frame bars. In such a case, the directional derivatives of Gâteaux are not the same as the Fréchet derivatives for a multiple eigenvalue, i.e., there is

$$\frac{\partial\lambda_i}{\partial h_p} \neq \mu_i(\mathbf{h}, \mathbf{e}_p), \qquad i = 1, 2, \quad p = 1, 2.$$

On the other hand, for the simple eigenvalues, the Gâteaux directional derivatives are equal to the Fréchet derivatives, i.e.:

$$\frac{\partial\lambda_i}{\partial h_p} \equiv \mu_i(\mathbf{h}, \mathbf{e}_p), \qquad i = 3, 4, \quad p = 1, 2.$$

Table 2 shows the directional derivatives with respect to the canonical basis vectors of the double eigenvalue of $\lambda_1 = \lambda_2$. The third column of Table 2 shows that in this case the additivity condition for the directional derivative of Gâteaux does not hold – which excludes the existence of Fréchet derivative at the point $\mathbf{h}$. The derivative matrix $\widehat{\mathbf{F}}^m$ is not diagonalizable by transformation $\boldsymbol{\Psi}_s^T\widehat{\mathbf{F}}^m\boldsymbol{\Psi}_k$, although matrix $\widehat{\mathbf{G}}^m$ is diagonalizable by transformation $\boldsymbol{\Psi}_s^T\widehat{\mathbf{G}}^m\boldsymbol{\Psi}_k$ – see Eqs. (3.6) and (3.8).

Table 2. Eigenvalues $\lambda_1 = \lambda_2$ and their directional derivatives along the vectors of canonical basis $\mathbf{e}_p$ of the parameter space $\mathbf{h} = [h_n, h_f]$.

| $\lambda_i$ | Deflected shape | $\sigma_i + \omega_i$ | $\mu_i(\mathbf{h}, \mathbf{e}_1)$ | $\mu_i(\mathbf{h}, \mathbf{e}_2)$ | $\mu_i(\mathbf{h}, \mathbf{e}_1 + \mathbf{e}_2) \neq \mu_i(\mathbf{h}, \mathbf{e}_1) + \mu_i(\mathbf{h}, \mathbf{e}_2)$ |
|---|---|---|---|---|---|
| $\lambda_1$ | antisym frame | −0.20516 +28.0212i | +0.01177 −0.80738i | −0.02963 +2.03316i | +0.00007 −0.00474i |
| $\lambda_2$ | sym beam | −0.20516 +28.0212i | −0.04451 +3.05466i | −0.00002 +0.00124i | −0.06246 +4.28642i |

Table 3 shows the directional derivatives with respect to the canonical basis vectors of the simple eigenvalues $\lambda_3$ and $\lambda_4$. In this case, the derivative along an arbitrary direction can be calculated from the scalar product of the gradient vector and the direction vector – as for the traditional Fréchet derivative.

Table 3. Eigenvalues $\lambda_3$ and $\lambda_4$ and their directional derivatives along the vectors of canonical basis $\mathbf{e}_p$ of the parameter space $\mathbf{h} = [h_n, h_f]$.

| $\lambda_i$ | Deflected shape | $\sigma_i + \omega_i$ | $\dfrac{\partial\lambda_i}{\partial h_n} \equiv \mu_i(\mathbf{h}, \mathbf{e}_1)$ | $\dfrac{\partial\lambda_i}{\partial h_f} \equiv \mu_i(\mathbf{h}, \mathbf{e}_2)$ | $\left[\dfrac{\partial\lambda_i}{\partial h_n} \quad \dfrac{\partial\lambda_i}{\partial h_f}\right] \circ (\mathbf{e}_1 + \mathbf{e}_2) \equiv \mu_i(\mathbf{h}, \mathbf{e}_1 + \mathbf{e}_2) = \mu_i(\mathbf{h}, \mathbf{e}_1) + \mu_i(\mathbf{h}, \mathbf{e}_2)$ |
|---|---|---|---|---|---|
| $\lambda_3$ | antisym beam | −1.69186 +80.6252i | −0.17604 +4.19511i | −0.03307 +0.78807i | −0.20910 +4.98318i |
| $\lambda_4$ | sym beam | −6.37071 +156.392i | −0.27743 +3.40018i | −0.11141 +1.36540i | −0.38884 +4.76550i |

## 7. Conclusions

The attractiveness and advantages of the presented approach lie in the consistent and clear matrix notation, which significantly facilitates the software application. The disadvantage, on the other hand, is the need to simultaneously determine both the derivatives of the eigenvectors and to redetermine the derivatives of the eigenvalues due to the coupling of the two equations in Eq. (4.11).

In the algorithm for calculating only the derivatives of the eigenvalues, the derivatives of the vectors need not be determined at the same time. In other algorithms presented in the literature (see the review of methods in, e.g., (Choi *et al.*, 2004)), leading directly to the derivatives of the eigenvectors of $\mathbf{\Gamma}_j$, it is not necessary to determine $\mu_j$ at the same time – nevertheless, many of these algorithms require the determination of all modal vectors of the eigenproblem in order to express $\mathbf{\Gamma}_j$ as their linear combination. The algorithm presented here enables the selection of the eigenmodes and frequencies for which the derivatives are calculated. Moreover, the redetermination of the eigenvalue derivatives is due to the fact that the combinations of the eigenvectors $\check{\mathbf{\Psi}}^m$ and $\check{\mathbf{\Phi}}^m$ are present in Eq. (2.19), which uses the coefficients $\boldsymbol{\theta}_j^m$ being the eigenvectors for each eigenvalue derivative $\mu_j$ – Eq. (3.10).

In fact, Eq. (3.10) has a low dimension in the practice of modelling structures in FEM due to the algebraic multiples of the multiple eigenvalue $\check{\lambda}_m$. For example, for rotationally symmetric shell structures, the eigenvectors are associated with the occurrence of double eigenvalues $n_a(\check{\lambda}_m) = 2$ – this implies the need to solve Eq. (3.10) with a matrix dimension [2×2]; however, rarely, with an exceptional arrangement of the structural parameters of these shells, the dual eigenvalues coincide, resulting in an algebraic multiplicity $n_a(\check{\lambda}_m) = 4$. Such a situation may lead to multiple derivatives as eigenvectors $n_a(\mu_j) > 1$ in Eq. (3.10).

The analogous case is shown in the paper – the frame has double eigenfrequencies $n_a(\check{\lambda}_m) = 2$ related to antisymmetric and symmetric mode shapes. It is a typical case in civil engineering for flat and spatial frames. The example presented in the paper shows that the issue of derivation of derivatives for multiple eigenvalues must be conducted in a non-traditional way in general cases. This entails also the need to develop specialized software and introduce additional conditions to distinguish multiple and single eigenvalues.

## References


1. Andrew, A.L., Chu, K.-W.E., & Lancaster, P. (1993). Derivatives of eigenvalues and eigenvectors of matrix functions. *SIAM Journal on Matrix Analysis and Applications*, *14*(4), 903–926. https://doi.org/10.1137/0614061
2. Balakrishnan, A.V. (1976). *Applied functional analysis. Applications of mathematics 3*. Springer.
3. Choi, K.-M., Jo, H.-K., Kim, W.-H., & Lee, I.-W. (2004). Sensitivity analysis of non-conservative eigensystems. *Journal of Sound and Vibration*, *274*(3–5), 997–1011. https://doi.org/10.1016/S0022-460X(03)00660-6
4. Day, D.M., & Walsh, T.F. (2007). *Quadratic eigenvalue problems*. Technical Report SAND2007-2072, Sandia National Laboratories. https://www.osti.gov/servlets/purl/912651
5. Garcia, S.R., & Horn, R.A. (2017). *A second course in linear algebra*. Cambridge Mathematical Textbooks. Cambridge University Press.
6. Gekeler, E.W. (2008). *Mathematical methods for mechanics. A handbook with MATLAB experiments*. Springer-Verlag, Berlin.
7. Hammarling. S., Munro, Ch.J., & Tisseur, F. (2013). An algorithm for the complete solution of quadratic eigenvalue problems. *ACM Transactions on Mathematical Software*, *39*(3), Article 18. https://doi.org/10.1145/2450153.2450156
8. Horn, R.A., & Johnson, Ch.R. (2013). *Matrix analysis* (2nd ed.). Cambridge University Press, New York.

9. Kim, D.-O., Kim, J.-T., Oh, J.-W., & Lee, I.-W. (1999a). Natural frequency and mode shape sensitivities of non-proportionally damped systems: Part I, Distinct natural frequencies. *Journal of the Computational Structural Engineering Institute of Korea*, *12*(1), 95–102.

10. Kim, D.-O., Kim, J.-T., Park, S.-K., & Lee, I.-W. (1999b). Natural frequency and mode shape sensitivities of non-proportionally damped systems: Part II, Multiple natural frequencies. *Journal of the Computational Structural Engineering Institute of Korea*, *12*(1), 103–109.

11. Krog, L.A., & Olhoff, N. (1995). Topology optimization of plate and shell structures with multiple eigenfrequencies. In N. Olhoff, & G.I.N. Rozvany (Eds.), *Structural and multidisciplinary optimization: Proceedings of The First World Congress of Structural and Multidisciplinary Optimization* (pp. 675–682). Pergamon Press.

12. Lancaster, P. (2013). Stability of linear gyroscopic systems: A review. *Linear Algebra and Its Applications*, *439*(3), 686–706. https://doi.org/10.1016/j.laa.2012.12.026

13. Lancaster, P., & Zaballa, I. (2009). Diagonalizable quadratic eigenvalue problems. *Mechanical Systems and Signal Processing*, *23*(4), 1134–1144. https://doi.org/10.1016/j.ymssp.2008.11.007

14. Lee, I.-W., Kim, D.-O., & Jung, G.-H. (1999a). Natural frequency and mode shape sensitivities of damped systems: Part I, Distinct natural frequencies. *Journal of Sound and Vibration*, *223*(3), 399–412. https://doi.org/10.1006/jsvi.1998.2129

15. Lee, I.-W., Kim, D.-O., & Jung, G.-H. (1999b). Natural frequency and mode shape sensitivities of damped systems: Part II, Multiple natural frequencies. *Journal of Sound and Vibration*, *223*(3), 413–424. https://doi.org/10.1006/jsvi.1998.2130

16. Leung, A.Y.T. (1993). *Dynamic stiffness and substructures*. Springer–Verlag.

17. Łasecka-Plura, M. (2023). A comparative study of the sensitivity analysis for systems with viscoelastic elements. *Archive of Mechanical Engineering*, *70*(1), 5–25. https://doi.org/10.24425/ame.2022.144077

18. Łasecka-Plura, M. (2024). Comprehensive sensitivity analysis of repeated eigenvalues and eigenvectors for structures with viscoelastic elements. *Acta Mechanica*, *235*(8), 5213–5238. https://doi.org/10.1007/s00707-024-03967-2

19. Martinez-Agirre, M., & Elejabarrieta, M.J. (2011). Higher order eigensensitivities-based numerical method for the harmonic analysis of viscoelastically damped structures. *International Journal for Numerical Methods in Engineering*, *88*(12), 1280–1296. https://doi.org/10.1002/nme.3222

20. Olhoff, N., Krog, L.A., & Lund, E. (1995). Optimization of multimodal structural eigenvalues. In N. Olhoff, & G.I.N. Rozvany (Eds.), *Structural and multidisciplinary optimization: Proceedings of The First World Congress of Structural and Multidisciplinary Optimization* (pp. 701–708). Pergamon Press.

21. Phuor, T., & Yoon, G.H. (2023). Eigensensitivity of damped system with distinct and repeated eigenvalues by chain rule. *International Journal for Numerical Methods in Engineering*, *124*(21), 4687–4717. https://doi.org/10.1002/nme.7331

22. Seyranian, A.P., Lund, E., & Olhoff, N. (1994). Multiple eigenvalues in structural optimization problems. *Structural Optimization*, *8*(4), 207–227. https://doi.org/10.1007/BF01742705

23. Tisseur, F., & Meerbergen, K. (2001). The quadratic eigenvalue problem. *SIAM Review*, *43*(2), 235–286. https://doi.org/10.1137/S0036144500381988

24. Wang, P., & Dai, H. (2015). Calculation of eigenpair derivatives for asymmetric damped systems with distinct and repeated eigenvalues. *International Journal for Numerical Methods in Engineering*, *103*(7), 501–515. https://doi.org/10.1002/nme.4901

25. Xu, Z., & Wu, B. (2008). Derivatives of complex eigenvectors with distinct and repeated eigenvalues. *International Journal for Numerical Methods in Engineering*, *75*(8), 945–963. https://doi.org/10.1002/nme.2280

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