JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
62, 4, pp. 787-798, Warsaw 2024
https://doi.org/10.15632/jtam-pl/194401

DYNAMIC CALCULATION OF THE FIRE ZONE FOR ANTI-AIRCRAFT ARTILLERY

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The article presents a fast algorithm for target hit probability calculation in real time for the use in modern fire control systems (FCS). The idea behind this algorithm is to calculate the hit probability using the information about the projectile motion and the estimated motion of the tracked target. The first part of the article describes the analytical solution to the problem of hit probability calculation. The results obtained from the analytical method are then compared with a simulation method developed specifically for the analytical method verification. This comparison led to the conclusion that the presented analytical method is suitable for the use in modern FCS.

Keywords: exterior ballistics, equations of motion, projectile trajectory

1. Introduction

The analysis of requirements posed by modern fire control systems showed the need to design and implement an algorithm that enables the target hit probability calculation in real time. This is a big challenge considering the complexity of algorithms required to calculate such probability. Those will include: processing of data coming from sensors (radars, cameras, lidars etc.), estimating motion parameters of the target, calculating the trajectory of the projectile. In the literature, one can find different approaches to hit probability definitions and calculation methods. In (Kang *et al.*, 2016), the hit probability prediction for the anti-aircraft artillery is based on errors of muzzle velocity and the cant error related to the fire power. Liu and Shi (2022) presented a method that extended the notion of hit probability to calculstion of different damage levels by incorporating the Bayesian inference method. Different approaches to artillery effectiveness are discussed in Katsev (2018), where authors defined it as a function of dependent and independent errors related to weapon-target interaction and how much would be artillery shell lethality. They showed an "offline" method to estimate effectiveness based on number of rounds, variations of random errors, dispersion errors, target lethal area. A similar approach was presented in (Obradović *et al.*, 2023), where authors also included errors in meteorological conditions preparation. Our innovative method proposes an analytical solution for real time calculation of hit probabilities based on both the projectile motion model and estimated parameters of the tracked motion. In order to verify the correctness of the algorithm, we also prepared an application that enabled simulation of the projectile flight and target movement, taking into account possible disturbances that might occur, such as: dispersion of the initial velocity of the projectile, deviations of projectile parameters from nominal values, cannon barrel jerk, etc.

2. Analytical method for target hit probability calculation

The analytical method for target hit probability calculations essentially boils down to four main steps (assuming that coordinates of the hit point and time to hit t_{hit} are already calculated):

1. Calculate how the perturbations of initial conditions of differential equations describing projectile motion propagate along its trajectory. In order to do that, one needs to solve variational equation (2.15) which is presented in Subsection 2.1. The system of equations describing projectile motion, Jacobi matrix for the system (which is necessary to solve the variational equation) and a numerical method (Runge-Kutta) for solving the system of differential equations is presented in Subsection 2.2.
2. Having the solution for the variational equation and the covariance matrix with initial variances for the system of equations, one can apply a rule of error propagation in order to find the covariance matrix at a specified time – Eq. (2.23).
3. The last element needed for hit probability calculation is the covariance matrix related to the target motion parameters. The initial covariance matrix is based on the Kalman filter output for the chosen motion model of the target. The value of the covariance matrix at time t_{hit} is calculated, as in the previous point, by applying the rule of error propagation – Eq. (2.24).
4. Having covariance matrices for the projectile and the target, calculate the hit probability using the Monte Carlo method described in Subsection 2.3.

The steps briefly listed above are described in more details in the following subsections.

2.1. Variational equation and error propagation

The variational equation describes how the disturbances of the initial conditions of the system will evolve along its trajectory[1]. There are several mathematical models commonly used to describe the motion of a projectile: the point-mass trajectory model (McCoy, 1999, the rigid body trajectory model (Baranowski, 2013a), the modified point mass trajectory model (Baranowski, 2013a).

Given any system of equations $\dot{\mathbf{y}}(t) = \mathbf{F}(\mathbf{y}(t), t)$ with the initial conditions $\mathbf{y}(t_0)$, one can easily generate a trajectory using numerical methods for iterative solution of ordinary differential equations (e.g. Runge-Kutta method). Te main goal in this part will be to determine how the solution of a system of equations behaves when a small disturbance $\boldsymbol{\delta}$ is introduced into the vector of initial conditions $\mathbf{y}(t_0)$.

Let

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = \mathbf{F}(\mathbf{y}(t), t) \tag{2.1}$$

be a system of n-th order differential equations with an initial condition $\mathbf{y}(t_0) = \mathbf{y}_0$ and a solution $\mathbf{\Phi}_t(\mathbf{y}_0, t_0)$, i.e. (Parker and Chua, 1989)

$$\dot{\mathbf{\Phi}}_t(t_0, \mathbf{y}_0) = \mathbf{F}(\mathbf{\Phi}_t(t_0, \mathbf{y}_0), t) \qquad \mathbf{\Phi}_{t_0}(t_0, \mathbf{y}_0) = \mathbf{y}_0 \tag{2.2}$$

Moreover, let

$$\boldsymbol{\delta} = \mathbf{\Phi}_{t_0}(t_0, \mathbf{y}_0) - \mathbf{\Phi}_{t_0}(t_0, \mathbf{y}_0 + \boldsymbol{\delta}) \tag{2.3}$$

be a disturbance of the initial conditions of the system and

$$\Delta\mathbf{\Phi}_t(t_0, \mathbf{y}_0) = \mathbf{\Phi}_t(t_0, \mathbf{y}_0) - \mathbf{\Phi}_t(t_0, \mathbf{y}_0 + \boldsymbol{\delta}) \tag{2.4}$$

[1] A problem often used when studying the stability of dynamical systems (Jordan and Smith, 2007)

be a result of the disturbance propagation after time t. Using the Taylor series expansion with the simultaneous omission of terms of higher orders (thus only the linear terms of the expansion are taken into account), one can write (2.4) as a function of the original disturbance

$$\mathbf{\Phi}_t(t_0,\mathbf{y}_0+\boldsymbol{\delta}) = \mathbf{\Phi}_t(t_0,\mathbf{y}_0) + \frac{\partial\mathbf{\Phi}_t(t_0,\mathbf{y}_0)}{\partial\mathbf{y}_0}\boldsymbol{\delta} \tag{2.5}$$

and thus

$$\mathbf{\Phi}_t(t_0,\mathbf{y}_0+\boldsymbol{\delta}) - \mathbf{\Phi}_t(t_0,\mathbf{y}_0) = \frac{\partial\mathbf{\Phi}_t(t_0,\mathbf{y}_0)}{\partial\mathbf{y}_0}\,\boldsymbol{\delta}. \tag{2.6}$$

Using definitions (2.3) and (2.4)

$$\varDelta\mathbf{\Phi}_t(t_0,\mathbf{y}_0) = \frac{\partial\mathbf{\Phi}_t(t_0,\mathbf{y}_0)}{\partial\mathbf{y}_0}\varDelta\mathbf{\Phi}_{t_0}(t_0,\mathbf{y}_0) \tag{2.7}$$

Resulting equation (2.7) is called the variational equation. In order to investigate the propagation of disturbances, it is necessary to find a part of the equation

$$\frac{\partial\mathbf{\Phi}_t(t_0,\mathbf{y}_0)}{\partial\mathbf{y}_0} \equiv \nabla_{\mathbf{y}_0}\mathbf{\Phi}_t(t_0,\mathbf{y}_0) \tag{2.8}$$

By integrating equation (**??**) an integral equation will be obtained

$$\mathbf{\Phi}_t(t_0,\mathbf{y}_0) = \mathbf{\Phi}_{t_0}(t_0,\mathbf{y}_0) + \int\limits_{t_0}^{t}\mathbf{F}(\mathbf{\Phi}_\tau(\tau_0,\mathbf{y}_0),\tau)\;d\tau \tag{2.9}$$

Differentiating with respect to initial values $\mathbf{y}_0$, we obtained

$$\nabla_{\mathbf{y}_0}\mathbf{\Phi}_t(t_0,\mathbf{y}_0) = \frac{\partial\mathbf{\Phi}_{t_0}(t_0,\mathbf{y}_0)}{\partial\,\mathbf{y}_0} + \int\limits_{t_0}^{t}\frac{\partial\mathbf{F}(\mathbf{\Phi}_\tau(\tau_0,\mathbf{y}_0),\tau)}{\partial\,\mathbf{y}_0}\;d\tau \tag{2.10}$$

Using the chain rule, and bearing in mind that

$$\frac{\partial\mathbf{\Phi}_{t_0}(t_0,\mathbf{y}_0)}{\partial\mathbf{y}_0} = \mathbb{I}_n \tag{2.11}$$

we obtained

$$\nabla_{\mathbf{y}_0}\,\mathbf{\Phi}_t(t_0,\mathbf{y}_0) = \mathbb{I}_n + \int\limits_{t_0}^{t}\frac{\partial\mathbf{F}(\mathbf{\Phi}_\tau(\tau_0,\mathbf{y}_0),\tau)}{\partial\mathbf{\Phi}_\tau(t_0,\mathbf{y}_0)}\frac{\partial\mathbf{\Phi}_\tau(t_0,\mathbf{y}_0)}{\partial\,\mathbf{y}_0}\;d\tau \tag{2.12}$$

By introducing the following notation

$$\mathbf{K}(t) = \nabla_{\mathbf{y}_0}\,\mathbf{\Phi}_t(t_0,\mathbf{y}_0) \qquad\qquad \mathbf{J}(t) = \frac{\partial\mathbf{F}(\mathbf{\Phi}_t(t_0,\mathbf{y}_0),t)}{\partial\mathbf{\Phi}_t(t_0,\mathbf{y}_0)} \tag{2.13}$$

and then returning to the differential form of Eq. (2.12), we finally got

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\nabla_{\mathbf{y}_0}\mathbf{\Phi}_t(t_0,\mathbf{y}_0)\right) = \frac{\partial\mathbf{F}(\mathbf{\Phi}_t(t_0,\mathbf{y}_0),t)}{\partial\mathbf{\Phi}_t(t_0,\mathbf{y}_0)}\frac{\partial\mathbf{\Phi}_t(t_0,\mathbf{y}_0)}{\partial\,\mathbf{y}_0} \tag{2.14}$$

then

$$\dot{\mathbf{K}}(t) = \mathbf{J}(t)\mathbf{K}(t) \qquad\qquad \mathbf{K}(t_0) = \mathbb{I}_n \tag{2.15}$$

here $\mathbf{J}(t)$ is a Jacobi matrix for the system of differential equations (2.3) at time t.

2.2. Error propagation for the M-model

The considerations presented in the previous subsection will be used to investigate the propagation of disturbances in the initial conditions for the explicit form of the modified point mass trajectory model, in (Baranowski, 2016b) called the M-model. This model is equivalent to the MPMTM model presented in (McCoy, 1999) and (STANG 4355, 2009) as demonstrated in (Baranowski *et al.*, 2016b)

$$\begin{aligned}
&\dot{x} = v_x + w_x \qquad \dot{y} = v_y + w_y \qquad \dot{h} = v_h + w_h \\
&\dot{p} = \frac{\rho v^2}{2I_x} S d C_{spin} \widehat{p} \qquad S = \frac{\pi d^2}{4} \\
&\dot{v} = -\frac{\rho v^2}{2m} S \left(C_{D_0} + \widehat{C}_{D^2_\alpha} \Big(\frac{2mg}{\rho v^2 S}\Big)^2 \frac{\widehat{I}_x^2 \widehat{p}^2 \cos^2\gamma_a}{(1-\widehat{I}_x\widehat{p}^2\widehat{C}_{mag-f})^2 + (\widehat{I}_x\widehat{p}\widehat{C}_{L\alpha})^2} \right) - g\sin\gamma_a \\
&\begin{bmatrix} \dot{\gamma}_a \\ \dot{\chi}_a\cos\gamma_a \end{bmatrix} = -\frac{g}{v} \frac{\cos\gamma_a}{(1-\widehat{I}_x\widehat{p}^2\widehat{C}_{mag-f})^2 + (\widehat{I}_x\widehat{p}\widehat{C}_{L\alpha})^2} \begin{bmatrix} 1-\widehat{I}_x\widehat{p}^2\widehat{C}_{mag-f} \\ \widehat{I}_x\widehat{p}\widehat{C}_{L\alpha} \end{bmatrix}
\end{aligned} \tag{2.16}$$

where: $\mathbf{x} = [x, y, h]$ – 3-dimensional position vector, $\mathbf{v} = [v_x, v_y, v_h]$ – velocity of the projectile with respect to the air, $\mathbf{w} = [w_x, w_y, w_h]$ – wind velocity vector, p – angular velocity of the spinning motion, ρ – air density, m – mass of the projectile, $\widehat{p} = pd/v$ - dimensionless coefficient, S – cross-section area of the projectile, C_{spin} – spinning drag coefficient, I_x – moment of inertia along the x axis, d – caliber, C_{D_0} – zero-yaw drag coefficient, g – gravitational acceleration, γ_a – the elevation angle of $\mathbf{v}$ measured from the horizontal direction, i.e. the air-path inclination angle, χ_a – the azimuth angle of $\mathbf{v}$, i.e. the air-path azimuth angle, $\widehat{C}_{mag-f}$ – dimensionless Magnus force coefficient, $\widehat{C}_{L\alpha}$ – dimensionless linear lift force coefficient, $\widehat{C}_{D^2_\alpha}$ – dimensionless yaw drag coefficient. The dimensionless coefficients are defined as

$$\widehat{C}_{D^2_\alpha} = \frac{C_{D^2_\alpha}}{C_{M_\alpha}} \qquad \widehat{C}_{L\alpha} = \frac{C_{L\alpha}}{C_{M_\alpha}} \qquad \widehat{C}_{mag-f} = \frac{C_{mag-f}}{C_{M_\alpha}} \tag{2.17}$$

where C_{M_α} – overturning moment coefficient, $C_{D^2_\alpha}$ – yaw drag coefficient, $C_{L\alpha}$ – lift force coefficient, C_{mag-f} – Magnus force coefficient.

In order to solve variational equation (2.15), it will be necessary to compute the Jacobi matrix for the explicit form of the modified point mass trajectory model presented above

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial F_1}{\partial \mathbf{y}_1} & \cdots & \dfrac{\partial F_1}{\partial \mathbf{y}_7} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_7}{\partial \mathbf{y}_1} & \cdots & \dfrac{\partial F_7}{\partial \mathbf{y}_7} \end{bmatrix}. \tag{2.18}$$

Using the Runge-Kutta method of the 4th order, one can find a solution $\mathbf{\Phi}_t(\mathbf{y}_0, t_0)$ of the system of equations (2.16), which will then enable us to calculate the value of Jacobi matrix (2.18) at each of the nodal points. The Runge-Kutta method is applied again to linear non-stationary system (2.15) in order to propagate the Jacobi matrix along the solution $\mathbf{\Phi}_t(\mathbf{y}_0, t_0)$ (Randez, 1992)

$$\begin{aligned}
&k_1^n = \mathbf{J}^{n-1} \qquad k_2^n = \mathbf{J}_H\Big(\mathbb{I} + \frac{h}{2}k_1^n\Big) \qquad k_3^n = \mathbf{J}_H\Big(\mathbb{I} + \frac{h}{2}k_2^n\Big) \qquad k_4^n = \mathbf{J}^n(\mathbb{I} + hk_3^n) \\
&\mathbf{K}_n = \Big(\mathbb{I} + \frac{h}{6}[k_1^n + 2(k_2^n + k_3^3) + k_4^n]\Big)\mathbf{K}_{n-1}
\end{aligned} \tag{2.19}$$

where $\mathbf{J}_H$ is the Jacobi matrix obtained as a result of interpolation between the nodal points $n-1$ and n, while the letter h is the integration step in the Runge-Kutta method.

2.3. Probability of hitting the target with an artillery projectile

Let the initial condition of equations (2.16) be described by a normal distribution

$$\mathbf{y}_0 \sim \mathcal{N}(\boldsymbol{\mu}_{y0}, \boldsymbol{\Sigma}_{y0}) \tag{2.20}$$

where

$$\boldsymbol{\mu}_{y0} = [\mu_{x0}, \mu_{y0}, \mu_{h0}, \mu_{p0}, \mu_{v0}, \mu_{\gamma_{a,0}}, \mu_{\chi_{a,0}}]^{\mathrm{T}} \tag{2.21}$$

is a vector of averages of the initial values of the system, and

$$\boldsymbol{\Sigma}_{y0} = \begin{bmatrix} \sigma_x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_y^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_h^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_v^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{\chi_a}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\gamma_a}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 \end{bmatrix} \tag{2.22}$$

is a matrix containing the variances of the initial position of the system. By using the rule of error propagation (Ochoa and Belongie, 2006), covariance matrix for the position of the considered system after time t will be

$$\boldsymbol{\Sigma}_{y(t_{hit})} = \mathbf{K}(t_{hit})\boldsymbol{\Sigma}_{y_{t_0}}\mathbf{K}^{\mathrm{T}}(t_{hit}) \tag{2.23}$$

where t_{hit} is the time in which the projectile travels to the point where it meets the target.

The respective variances in (2.22) refer to: projectile position in three dimensional space, its velocity, elevation and azimuth angles, rotational speed. At this point, it is necessary to discuss some assumptions that were adopted when solving the problem for the selected model of the projectile flight:

1. The physical parameters of the projectile, such as mass and caliber, are treated as constant values.
2. The quantities used in Eq. (2.22) are described with the normal distribution where:
   - the initial position will depend on the measuring device used to establish the initial position;
   - parameters of speed distribution and dispersion of the gun in the elevation and azimuth angle planes are provided by the manufacturer;
   - the parameters of the distribution of the rotational speed of the projectile result directly from the parameters of the distribution of its muzzle velocity.

The covariance matrix related to the position of the tracked target at the meeting point has yet to be found. The value of this matrix will be calculated using the error propagation law

$$\boldsymbol{\Sigma}_{c(t_{hit})} = \mathbf{A}(t_{hit})\boldsymbol{\Sigma}_{c(t)}\mathbf{A}^{\mathrm{T}}(t_{hit}) \tag{2.24}$$

where $\mathbf{A}$ is the matrix of motion dynamics of the tracked target depending on the assumed target motion hypothesis (constant velocity motion, constant acceleration motion, constant turn motion etc.), $\boldsymbol{\Sigma}_{c(t)}$ is the covariance matrix related to the estimated (e.g. by the use of Kalman filtering) target motion parameters at the current moment t for the selected motion hypothesis, $\boldsymbol{\Sigma}_{c(t_{hit})}$ is the covariance matrix after the time t_{hit} for the selected motion hypothesis. Having the information about the target and projectile position in the area around the meeting point, the next step will be to calculate the probability of projectile and target collision. To solve this problem, the Monte Carlo method can be used, and the calculations are as follows:

1. Finding vectors and eigenvalues for the covariance matrices $\mathbf{\Sigma}_{y(t_{hit})}$ and $\mathbf{\Sigma}_{c(t_{hit})}$ such that

$$\mathbf{\Sigma}_{y(t_{hit})} = \mathbf{V}_p\mathbf{D}_p\mathbf{V}_p^{\mathrm{T}} \qquad \mathbf{\Sigma}_{c(t_{hit})} = \mathbf{V}_c\mathbf{D}_c\mathbf{V}_c^{\mathrm{T}} \tag{2.25}$$

   where $\mathbf{D}_p$ and $\mathbf{D}_c$ are diagonal matrices containing eigenvalues of the covariance matrix for the calculated projectile and target position respectively, $\mathbf{V}_p$ and $\mathbf{V}_c$ – matrices whose columns contain the matrix eigenvectors for the calculated projectile position and target, respectively.

2. Drawing two sets of points in the three-dimensional Cartesian space

$$\mathbf{X}_c = [\mathbf{x}_c, \mathbf{y}_c, \mathbf{h}_c]\mathbf{D}_c \qquad \mathbf{X}_p = [\mathbf{x}_p, \mathbf{y}_p, \mathbf{h}_p]\mathbf{D}_p\mathbf{R} \qquad \mathbf{R} = \mathbf{V}_c\mathbf{V}_p \tag{2.26}$$

   where $\mathbf{x}_c, \mathbf{y}_c, \mathbf{h}_c, \mathbf{x}_p, \mathbf{y}_p, \mathbf{h}_p$ are N-element column vectors drawn from a standardized normal distribution.

3. Calculate probability by summing the number of elements in both sets ($\mathbf{X}_c$ and $\mathbf{X}_p$) that are simultaneously in the target surrogate area, and then divide the result by N.

The probability of hitting the target, calculated according to the above algorithm, is determined with a certain accuracy and strictly depends on the selection of the surrogate area referred to in point 3 of the algorithm description.

3. Simulation method for finding the probability of hitting the target

The method of calculating the probability that will be presented in this s Section has been prepared as a reference tool. The main reasons why this method cannot be used in a real time calculations of the hit probability are:

- basing the algorithm on the assumption that the ideal trajectory of the tracked object is known;
- very long simulation time, the performance of which is necessary to determine the target hit probability.

This algorithm can be divided into two main parts, the first of which will be responsible for simulating the projectile flight in the atmosphere, while the second will focus on simulating the movement of the tracked target.

Projectile motion in the atmosphere will be described by a system of differential equations (2.16)

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{y}(t) = F_p(t, \mathbf{y}(t)) \tag{3.1}$$

with rhe initial condition

$$\mathbf{y}(t_0) = \mathbf{y}_0 \tag{3.2}$$

In real conditions, each quantity in vector (3.2) is burdened with a certain error, which will be described for the purposes of simulation tests with a normal distribution. In the conducted simulations, the vector of initial conditions will have the form

$$\mathbf{y}(t_0) = \mathbf{y}_0 + \mathit{\Delta}\mathbf{y}_0 \tag{3.3}$$

where the respective elements of the vector $\mathit{\Delta}\mathbf{y}_0$ are described with the use of normal distribution

$$\mathit{\Delta}{y_0}^i \sim \mathcal{N}(\mu_i, \sigma_i^2) \tag{3.4}$$

It should also be remembered that not only the initial conditions affect the projectile trajectory. The equations also use the projectile physical parameters (mass, diameter) denoted as $\mathbf{q}$, which are also burdened with a normally distributed error

$$\mathbf{q}_n = \mathbf{q} + \Delta\mathbf{q} \qquad \Delta q_j \sim \mathcal{N}(\mu_j, \sigma_j^2) \tag{3.5}$$

The values of distribution parameters for each quantity depend on the type of ammunition and are specified by the manufacturer. The flight trajectory simulations will be carried out in a standard atmosphere – uncertainties related to determination of the meteorological situation will be ignored. The reason for this approach is primarily the problem of obtaining appropriate data that can be used to define the aforementioned uncertainties. However, it should be emphasized that the prepared application allows one to take them into account as much as possible. The second essential part of the probability calculation algorithm is related to the simulation of the target movement being fought by the anti-aircraft system. In order to carry out simulation tests, sets of test trajectories were prepared with the use of the route generator.

Each of the sets included two types of routes: **ideal trajectory** with successive target positions generated with a selected frequency, treated at a later stage as a reference trajectory, **noisy trajectory** with subsequent measurements of angular coordinates and distance to the target – measurement errors and the frequency of measurements depends on the sensors used in the anti-aircraft set (radar, optoelectronic head, etc.). After each simulation, it is checked whether the projectile hits the tracked object with the given perturbations for the parameters and the initial conditions of the differential equations describing the projectile motion. The so-called standard NATO objectives being cuboids with dimensions:

- $12 \times 2.3 \times 2.3$ m represents fighter-type targets;
- $2 \times 0.5 \times 0.5$ m represents drone targets.

The application developed in the Matlab environment has the ability to simulate the described situation for various types of targets and ammunition together with visualization of the entire situation. This approach to calculating the probability is a very good verification tool, but requires a lot of time and computing power.

4. Comparison of the results for algorithms calculating the target hit probability

In order to verify the correctness of the algorithm of the analytical method for calculating the probability of hitting a tracked object with a single bullet, appropriate simulation tests were carried out. The research included calculating the hit probability using the analytical and simulation method (considered as the reference method) for the following variants of air target movement: stationary target, target approaching the artillery, moving with constant velocity motion, target moving away from the artillery with constant velocity motion. During the simulation tests, only uncomplicated types of target dynamics were considered – which are sufficient to evaluate the probability calculation method, which does not depend on the type of target movement. For all the cases described above, the centre of the coordinate system was assumed at the point where the artillery is standing. The generated ideal trajectory of the target was disturbed, for the purposes of research, with the following measurement errors:

- white noise has been added to the azimuth and elevation angles $\mathcal{N} \sim (0, \sigma_{a,e})$, where $\sigma_{a,e} = 0.3$ mrad;
- quantization error was added to the distance measurement $\Delta_r = 2$ m.

The trajectory of the projectile fired towards the meeting point will be calculated using the equations of the modified point mass trajectory model in its explicit form (see equations

(2.16)). Propagation of disturbances of the initial conditions of differential equations around the solution is performed according to the algorithm presented in Subsection 2.2, using Jacobian for the explicit form of the modified point mass trajectory model. The initial conditions disturbances used in the simulations have the following values:

- $\Delta x_0, \Delta y_0, \Delta h_0 \sim \mathcal{N}(0, \sigma_x)$, where $\sigma_x = 0.5\,\mathrm{m}$;
- $\Delta v_0 \sim \mathcal{N}(0, \sigma_v)$, where $\sigma_v = 1.5\,\mathrm{m/s}$;
- $\Delta\chi_{a,0} \sim \mathcal{N}(0, \sigma_{\chi a})$, where $\sigma_{\chi_a} = 0.32\,\mathrm{mrad}$;
- $\Delta\gamma_{a,0} \sim \mathcal{N}(0, \sigma_{\gamma_a})$, where $\sigma_{gamma_a} = 0.32\,\mathrm{mrad}$;
- $\Delta p_0 \sim \mathcal{N}(0, \sigma_p)$, where $\sigma_p = 0\,\mathrm{rad/s}$.

The above values of the initial conditions disturbance are used both in the analytical and reference methods for calculating the target hit probability.

4.1. Stationary target

The tests were carried out for different distances of the target from the firing position. The paper presents exemplary calculation results for the target located on the border of the effective range. In this case, it was assumed that the target is at point $C = (x_c, y_c, h_c) = (-4000, 300, 700)$ relative to the starting point of the projectile, which is also the center of the coordinate system. The target tracking process took 25 s, and the hit probability was calculated at $t_p = 1, 3, 5, \ldots, 19, 21\,\mathrm{s}$. The results of simulation tests for the discussed case are included in Table 1.

Table 1. The probability of hitting a stationary target calculated in subsequent moments of simulation. P_{H1} – probability calculated using the reference method, P_{H2} – probability calculated using the analytical method

	t_1 [s]										
	1	3	5	7	9	11	13	15	17	19	21
P_{H1}	0.570	0.568	0.550	0.564	0.556	0.574	0.534	0.594	0.570	0.550	0.572
P_{H2}	0.002	0.030	0.481	0.481	0.408	0.477	0.481	0.481	0.483	0.480	0.482

At this point, it is necessary to discuss the differences that occurred when calculating the probability of an artillery projectile hitting the tracked target. The most distinct difference occurred in the first second of the tracking process (probability values are highlighted in orange). In Fig. 1a, one can see how the trace of the covariance matrix resulting from the operation of the Ballistic Computing Module and related to the calculated future target position has changed. Only after 3 s, the filter went into a steady state – the appropriate target dynamics was adjusted, which increased the accuracy with which the target position was determined.

A stationary target close to the limit of the effective range of the tested type of artillery ammunition is, contrary to intuition, a case for which the determination of the motion parameters (and thus the prediction of the future location) is relatively difficult. From the point of view of the FCS, the target moves completely randomly depending on the measurement errors of sensors (in real conditions, often also on atmospheric conditions) - an examplary trajectory is shown in Fig. 1.

4.2. Constant velocity motion – incoming target

The starting point of the trajectory of the tracked target is $C_0 = (x_0, y_0, h_0) = (-4500, 300, 700)\,\mathrm{m}$; the target is moving with a constant velocity $\mathbf{V}_c = [v_x, v_y, v_h] = [100, 0, 0]\,\mathrm{m/s}$. The movement of the target is simulated over the time $t_s = 22\,\mathrm{s}$ (the target

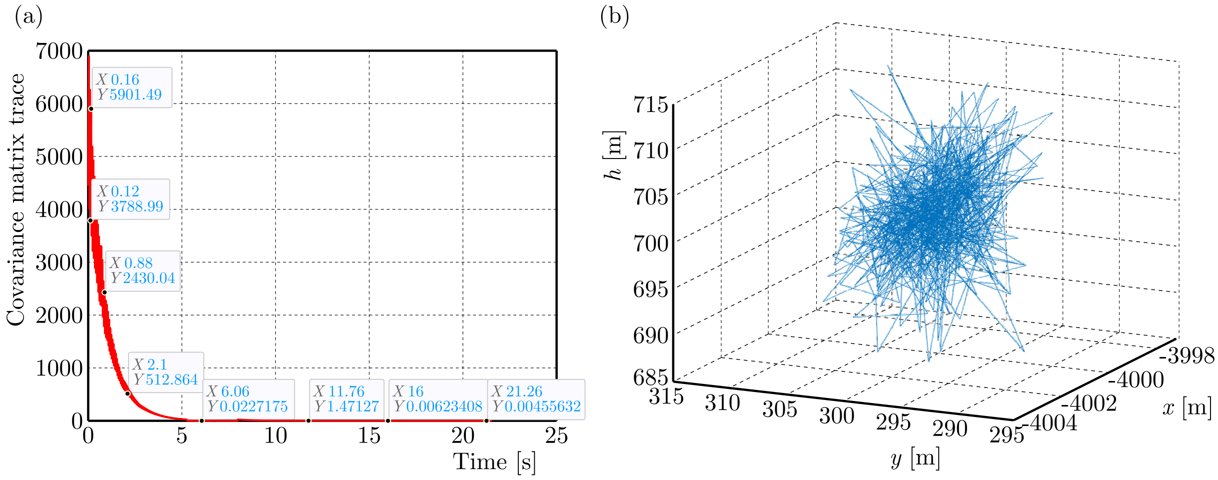

Fig. 1. (a) Trace of the covariance matrix resulting from the operation of BCM and related to the location of the tracked target (point $C = (-4000, 300, 700)$). The figure shows trace values for the covariance matrix at selected moments in the tracking process. (b) The trajectory of a stationary point recorded by the system after taking into account sensor measurement errors

trajectory is shown in Fig. 2b). The probabilities of hitting the target, calculated in the moments $t_p = 1, 3, 5, \ldots, 19, 21$ s are presented in Table 2. In the case of the analytical method, the values of target hit probability increase over time. This is understandable given the trace of the covariance matrix (Fig. 2a) related to the determination of the target location by the chosen filtering method. Compared to the stationary target, the target dynamics was selected by the system very quickly (less than 1 s), and because it was selected in accordance with the actual target movement, errors in determining target motion parameters decreased shortly after starting the tracking process.

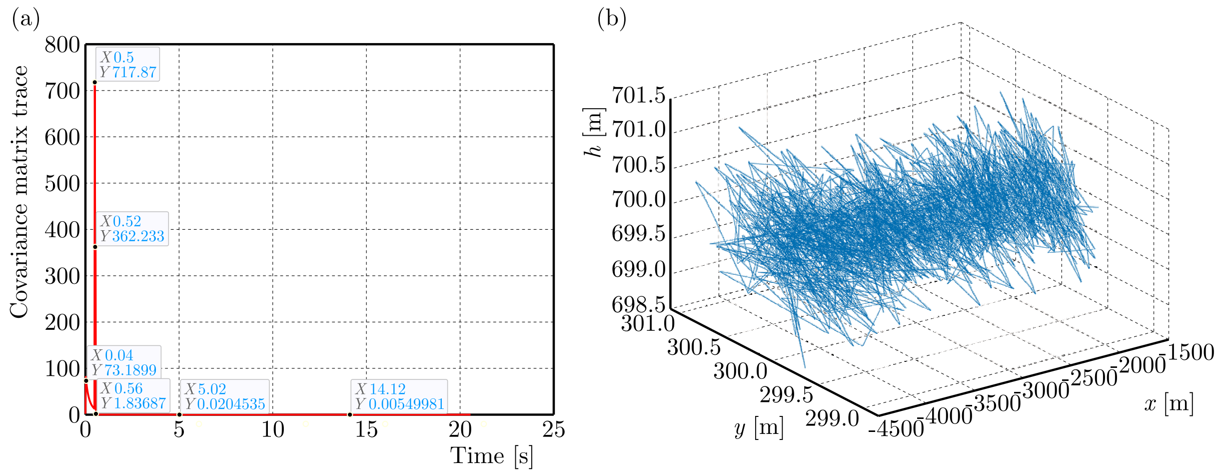

Fig. 2. (a) Trace of the covariance matrix resulting from the operation of FCS and related to the location of the tracked target (the target approaching the artillery gun with constant velocity motion). The figure shows the trace values of the covariance matrix at selected moments in the tracking process. (b) The trajectory of the target moving with the constant velocity motion model (the target is approaching) recorded by the system after taking into account sensor measurement errors

The analytical method is sensitive to the information coming from the FCS, and more precisely, to the values of the covariance matrix related to the parameters of the target movement. The stabilization of the tracking process with the Kalman filter causes that the set of random points representing the position of the target around the meeting point, Eq. $(2.25)_2$, is very

Table 2. The probability of hitting a target moving with a constant velocity, calculated in successive moments of the simulation. P_{H1} – probability calculated using the reference method, P_{H2} – probability calculated using the analytical method

	t_1 [s]										
	1	3	5	7	9	11	13	15	17	19	21
P_{H1}	0.418	0.628	0.670	0.686	0.706	0.768	0.798	0.862	0.912	0.910	0.894
P_{H2}	0.638	0.682	0.737	0.710	0.775	0.842	0.881	0.914	0.948	0.989	0.996

concentrated around that point. Moreover, in the case of an approaching target, the projectile trajectory to the meeting point, calculated in each subsequent iteration of the FCS, is shorter than the previous one. Therefore, it can be expected that as a result of the propagation of disturbances around the solution of the projectile motion equations, the obtained values of the variance of the projectile motion parameters decrease with each subsequent iteration of the FCS. This naturally translates into an increase in the value of the calculated probability.

Also in the case of the reference method, the values of the probability of hitting the target increase in the following moments. The reasons are similar to those given for the analytical method – approaching the target to the center of fire reduces the dispersion of projectiles around the meeting point and, at the same time, the gun settings used in the simulation of the projectile flight are determined by the FCS with a smaller error (resulting directly from the accuracy of determining the movement parameters of the tracked target).

In the case under consideration, the discrepancies between both methods of calculating the probabilities are greater. This indicates a slightly greater sensitivity of the analytical method to the information on accuracy of determining the target location coming from the FCS. One should also remember about a certain assumption that affects the probabilities between the two methods - the reference method is based on the assumption that the real trajectory of the target is known. On the other hand, in the analytical method, the location of the target is determined only on the basis of the estimated parameters of the target movement from the FCS.

4.3. Constant velocity motion – receding target

The initial position of the trajectory of the tracked target is $C_0 = (x_0, y_0, h_0) = (500, 300, 700)$ m; the target moves with a constant velocity $\mathbf{V}_c = [v_x, v_y, v_h] = [100, 0, 0]$ m/s (Fig. 3b). The target movement is simulated for $t_s = 22$ s. The probability of hitting the target, calculated at times $t_p = 1, 3, 5, \ldots, 19, 21$ s, is presented in Table 3.

Table 3. The probability of hitting a target moving with a constant velocity, calculated in successive moments of the simulation. P_{H1} – probability calculated using the reference method, P_{H2} – probability calculated using the analytical method

	t_1 [s]										
	1	3	5	7	9	11	13	15	17	19	21
P_{H1}	0.851	0.754	0.946	0.878	0.858	0.890	0.8680	0.830	0.798	0.816	0.824
P_{H2}	1.000	1.000	1.000	0.998	0.982	0.966	0.9170	0.852	0.795	0.731	0.679

In this case, the probability calculated by the analytical method decreases as the target moves away. As onr can see in Fig. 3a, the errors in determining the target motion parameters in the first moments of the tracking process are very small (especially compared to the errors occurring in the case of an approaching or stationary target). It is understandable, taking into account the fact that measurement errors of the azimuth and elevation angles propagate with the distance from the sensor. Fluctuations in the values of the probability determined with the use

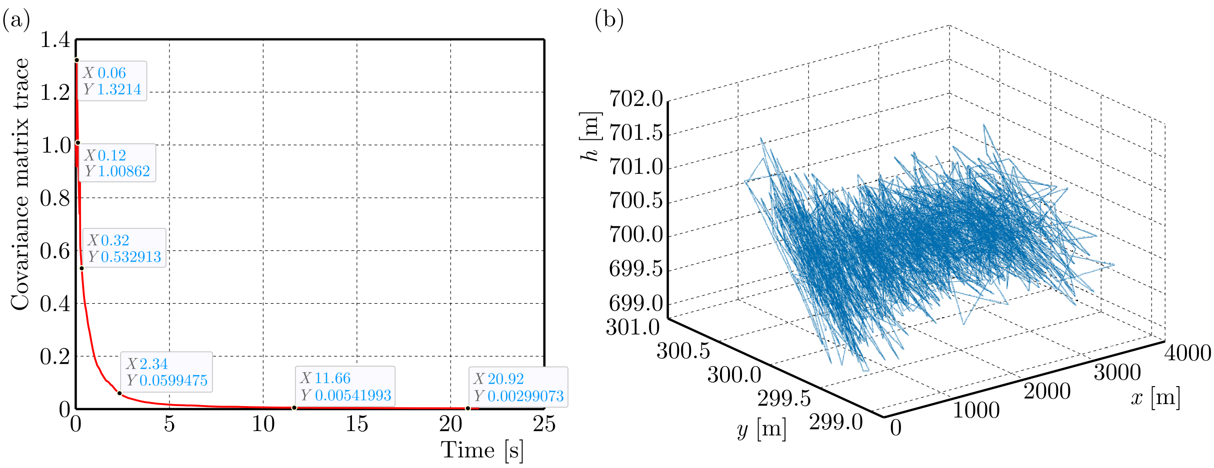

Fig. 3. (a) Trace of the covariance matrix resulting from the operation of FCS and related to the location of the tracked target (the target moving away from the fire center with a constant velocity). The figure shows the trace values of the covariance matrix at selected moments in the tracking process. (b) The trajectory of the target moving with a constant velocity (the target moves away) recorded by the system after taking into account sensor measurement error

of the reference method are caused by the sensitivity of the method to incorrectly determined gun settings at a given moment.

5. Conclusions

The algorithm for analytical calculation of the probability was verified using a simulation tool prepared in the Matlab environment based on the assumption that the real target trajectory is known. Significant discrepancies between the results obtained from both methods are a direct outcome of the filter bank implemented in the Fire Control System. As has been shown, the algorithms produce very similar results when the Kalman filter matched to a given target dynamics enters a steady state. Until this moment, the uncertainties in determining the target location around the meeting point are too high. It should be emphasized, however, that the time needed by the filter to reach the steady state is very short, up to 2 s.

The time of the filter transition to the steady state, unfortunately, cannot be predicted with an efficient method – it depends on many factors: measurement errors of sensors used for detecting objects, dynamics and motion parameters of the tracked target. An important factor in the tracking process are maneuvers performed by the tracked target, weather conditions that may have a negative impact on the correct operation of the sensors, e.g. the influence of rain on the operation of the laser rangefinder, the influence of clouds on the operation of the videotracker. The main advantage of the developed analytical method (apart from the aforementioned compatibility with the simulation tool) is its runtime. The time of one FCS iteration is 20 ms. The time allocated to the Ballistic Calculation Module is 14 ms. In the case of using 6 anti-aircraft guns, the maximum execution time of the probability calculation algorithm is 1 ms.

The algorithm operation time was tested on a ballistic computer used in the Fire Control System. Runtime of the algorithm is mainly influenced by three factors: chosen mathematical model for projectile motion, number of steps in the integration method of differential equations describing motion of a projectile, number of samples in the Monte Carlo simulation. It is obvious that reducing the number of steps and samples will shorten the algorithm runtime, but will negatively affect its accuracy. These values should be selected depending on the specification of the hardware by which the calculations will be performed. The analytical method for calculating the hit probability can be used as a tool to support the operator decision making. Firing a shot after reaching the appropriate hit probability level should reduce the amount of ammunition needed to neutralize the target being tracked. Fewer projectiles will translate into increased service life of both the barrel and the gas chamber of the anti-aircraft gun.

Acknowledgements

This work was financed by Military University of Technology under research project UGB 824/2023.

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Manuscript received September 25, 2023; accepted for publication October 9, 2024