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TWO TOOLS FOR THE SCIENCE OF MATERIAL EFFORT – A REVIEW PAPER[1]

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This paper discusses the science of material effort from the historical viewpoint. Two general scientific tools: the geometrical descriptive method of Mohr, and the energetic method of Huber are compared and evaluated from the very beginning. Three appropriate stress invariants are taken into account: stress intensity, stress triaxiality and stress shearness. Especially, much attention is devoted to explanation of the stress shearness invariant, which aims at describing the Lode parameter in a more analytical manner. Two different tools of finding a proper yield surface which contains the above mentioned three stress invariants are discussed in the literature perspective. In particular, the three-parameter yield surface, called the Burzyński-Pęcherski hypothesis is researched and explained from this new of point view.

Keywords: Tresca-Mohr hypothesis, Beltrami-Huber, Mises, Burzyński Hypothesis, Lode parameter, material effort

1. Introduction and motivation

1.1. Huber's science of material effort

In 1904, Tytus Huber established the science of material effort as a field of knowledge between three-dimensional theory of elasticity and zero-dimensional (engineering) science of the strength of materials. The science of effort was applied then to accomplish the most difficult thing – it was to use the language and mathematical methods of the three-dimensional theory of elasticity to determine the state of material effort by means of a single characteristic quantity called "effort". This quantity, measured as volumetric energy density [J/m^3], was denoted by Huber with the letter W (from the Polish word "wytężenie").

Why did Huber decide to determine the effort state of a material by a single characteristic scalar $W(t, \mathbf{x})$? Why did he choose such a simple measure of effort? Why did he not take, for instance, three invariants of the stress tensor $I_1 = \sigma_{ii}$, $J_2 = (1/2)s_{ij}s_{ij}$, $J_3 = (1/3)s_{ij}s_{jk}s_{ji}$ as three measures of effort?

Huber assumed that the effort state of the material must be expressed by a single scalar quantity, since the science of effort had been supposed to combine mathematical theory with

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engineering practice, and engineering practice at that time used a single critical quantity, which was the "maximum strength". In order to compare that integral, 0-dimensional quantity, measured at the level of the entire device, with the effort $W(t, \mathbf{x})$ determined locally at the continuum point level. Huber (1904) introduced the concept of critical energy density and denoted it with the letter K [J/m^3]. It no longer depended on a point in the material, as it had been defined from measurements on a large sample. This was when the first mathematical expression of the science of material effort arose

$$W(t, \mathbf{x}) \leqslant K \tag{1.1}$$

It was spoken as a rule: the material effort must not exceed the strength of the material. The engineers adopted this principle as a design principle to "effort yourself to endure" (in Polish: wytężać tak, aby wytrzymywać). In his work, Huber develops three different definitions of the measure of the material effort. To prove which ones are better, he compares the results with experimental results and with the predictions resulting from Otto Mohr's geometrical approach (Mohr, 1882). At that time, geometrical methods were considered superior to analytical methods. It must be remembered what was the dominant status of "methods of descriptive geometry" in science and mechanics at that time. An analytical model that did not agree with Mohr's geometrical results was considered inferior or even conceptually wrong by early 20th-century scholars. A dispute arose among theoreticians as to which of the approaches was the right one: Huber's or Mohr's? In other words, which of the two research tools – geometrical or energetic – is correct? A challenge arose among experimenters to prove experimentally which of the approaches was valid? Stepan Timoshenko (1953) mentions and discusses the key experiments of the period 1900-1930, among them the Lode experiment (1926).

1.2. Mohr's science of material strength

Mathematically, Mohr's geometric approach was also innovative. It referred to the method proposed by Clapeyron and Lame in 1832 and Stokes in 1845 to construct constitutive equations only through principal values of the stress tensor and principal values of the strain tensor, respectively. The most important thing in that method was to find the principal stresses σ_1, σ_2, σ_3 using the descriptive method (the so-called Mohr circle; Mohr, 1882). And then, in the corresponding diagram, τ-σ making the so-called maximum envelopes of the circles (Fig. 1).

Let us emphasize, the geometrical method led Mohr to the hypothesis of marginal envelopes, which does not belong to any of the three types of hypotheses – it is not the hypothesis of extreme stresses, extreme deformations, or the hypothesis of extreme energies. It is a geometrical hypothesis developed for the needs of bulk materials such as sand or grain. Perhaps, the envelope hypothesis, so strongly related to the construction of Mohr's circle, is some kind of "circling" around a new analytic expression for effort. Therefore, when we begin to study Mohr's hypothesis, we should keep in mind Burzyński's question "Are Mohr's envelopes the complete content of his hypothesis of material effort, or are they an illustration of some unknown hypothesis, being a mere means of drawing, more or less successful?"

In 1900, Otto Mohr decided to introduce a new concept: the limit shear strength k_{tr}, which had a weak experimental reference (Mohr, 1900). This means that among the conceivable deformation states, in addition to the torsional state, there is a condition that shows similarity to technological cutting. Mohr explains this limit state as an independent state, always occurring in addition to tension, compression and torsion, being the fourth limit state – Mohr hypothesizes that this state can be determined by shear stresses and that the maximum limit value of k_{tr} occurs in a cross-section in which the normal stress is equal to zero. Hence Mohr's magnitude: k_{tr} he calls "shear strength" – but this is a concept difficult to grasp experimentally. Hence, this notion is criticized by, for instance, A. Föppl in his monograph: *Exercises in Technical Mechan-*

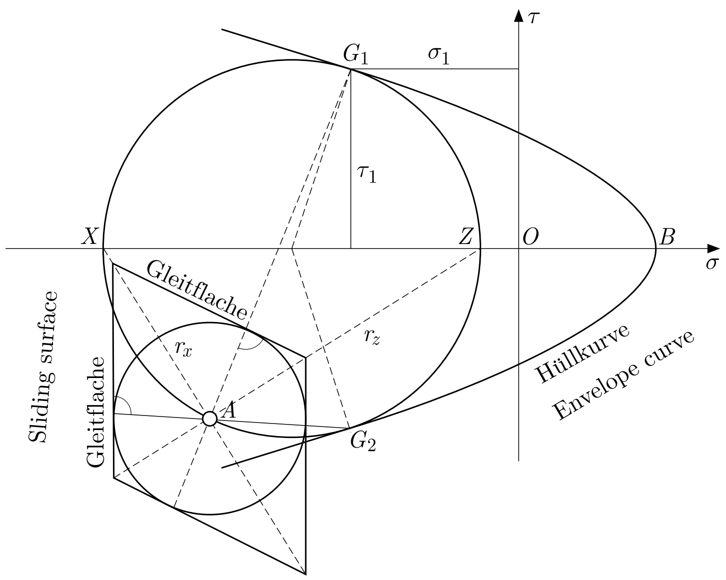

Fig. 1. Mohr's envelope plot in the plane σ-τ (Mohr, 1906, Fig. 13)

ics, from 1907 (Föppl, 1907). The point is how to express it: k_{tr} through the measurement-known limits of k_t, k_c.

Undoubtedly, the starting point of Mohr's reasoning was the Tresca hypothesis, which referred to the maximum shear stress and the parameter k_s that determines it. It can also be called a constant marginal shear stress hypothesis (Pełczyński, 1958). It is defined in the principal axes as $\sigma_{eq,t}^{Tresca} = (\sigma_1 - \sigma_3)/2 \leqslant k_s$ where the stress σ_{eq}^{Tresca} is the stress equivalent to the shear test from which we obtain the value of k_s. If we assume, as it is "done" by many researchers, that $k_s = k_t/2$, then the Tresca stress equivalent to the uniaxial tensile test is the equation $\sigma_{eq,r}^{Tresca} = \sigma_1 - \sigma_3 \leqslant k_r$.

Then, in the Mohr plane τ-σ, one can draw the boundaries of the Mohr circles that have radius $(\sigma_1 - \sigma_3)/2$, and this boundary is two lines parallel to σ. On the other hand, assuming that $k_t = k_c$, then in the space of principal stresses, we can assign a prism with a symmetrical hexagonal cross-section to the above condition.

As Huber mentions in his historical outline of the development of the effort hypotheses (*Technical Stereomechanics*, p. 88, Huber, 1948), Mohr's motivation was to consider a combined shear in the slip plane τ and the internal friction forces proportional to the pressure force σ in this cross-section. With this motivation in front of him, Mohr writes the effort condition as $\sigma_{eq}^{CM} = \tau_{max} = \tau + f\sigma \leqslant k_{tr}$.

It is therefore an adaptation of the hypothesis of extreme internal friction developed by Coulomb in 1776 known as the hypothesis *slippage with friction* $\pm\tau + f\sigma \leqslant k_{tr}$. It is a hypothesis belonging to the group of hypotheses based on two constants: the coefficient of friction f and the cohesive force k_{tr}. It was developed by L. Navier (1837) and Ch. Duguet (1885); especially for loose and porous media. Therefore, the critical state is reached as a result of the formation of slip planes, which, thanks to the forces of internal friction and cohesion, are inclined in relation to the main axes at angles slightly different from $\pi/4$ and for metals, according to Duguet, it is about $\beta = 10°$, which gives the coefficient of friction $f = \tan\beta = 0.176$ (Timoshenko, 1953).

Let us remember that other concepts of the material effort, such as Mohr's or Burzyński's, have given up on the constancy of the f coefficient. In Coulomb's hypothesis, the envelopes of Mohr's circles are instantaneous. Moving now from the constants f and k_{tr} to the available experimental data, we can see that at least two types of experiments are needed to determine them – for example, a uniaxial tensile test and a uniaxial compression test. Then we first write the above condition as (Pełczyński, 1959)

$$\frac{\sigma_1-\sigma_3}{2}+\frac{\sigma_1+\sigma_3}{2}n \leqslant k_{tr} \tag{1.2}$$

We have gone here permanently, $n=\sin\beta$ and $k_{tr}=k_t\cos\beta$. In turn, these constants are easy to express in terms of tensile and compressive limit constants k_t and k_c as

$$n=\frac{k_c-k_t}{k_c+k_t} \qquad\qquad k_{tr}=\frac{k_c k_t}{k_c+k_t} \tag{1.3}$$

This means that we do not have to measure the angle of internal friction and the constant of cohesion for metals, but it is enough to perform two uniaxial tests, compression and tensile. When the limits of $k_t=k_c$ are equal, the angle of internal friction disappears, and Coulomb's hypothesis comes down to Tresca's hypothesis.

A fundamental feature of Mohr's exertion model has gone down in the history of the problem, namely that it does not take into account the mean (second) principal stress σ_2 while Huber's hypothesis does. Further reconstructions of Mohr's approach are provided in Dudda's (2021) paper.

1.3. Lode's crucial experiment

It was Walther Lode who undertaken the question which hypothesis – Huber's or Mohr's – is correct (Lode, 1926). Lode having only 11 probes (5 cast iron, 5 cooper and 1 nickel) in a form of long thin-walled pipes, performed tests loaded simultaneously by internal pressure and uni-axial tension or compression. Since, in that case, all three invariants σ_1, σ_2, σ_3 were easy to be determined, Lode was able to proof the influence of the second main stress σ_2 on strength limits. Denoting by μ (now the Lode parameter), some stress shearness losses he expressed second principal stress as $\sigma_2=0.5(\sigma_1+\sigma_3)+0.5\mu(\sigma_1-\sigma_3)$. Knowing σ_1, σ_2, σ_3 from the experimental data, he defined the value of μ from (Lode, 1926, his Eq. 23)

$$\mu=\frac{2\sigma_2-\sigma_1-\sigma_3}{\sigma_1-\sigma_3} \qquad \text{with} \qquad \sigma_1\geqslant\sigma_2\geqslant\sigma_3 \tag{1.4}$$

Therefore, when the state of pure shear is realized, the Lode parameter is to be zero, what means that in that state the Tresca and Coulomb-Mohr criterions are best. But in a real state of stresses, the prefect shearness is a lit bit loosed, therefore $\mu\neq 0$. By a simple inspection Lode finds that $\mu=-1$ in the state of uniaxial tension or two-axial compression, and $\mu=1$ for uniaxial compression and double-tension. Next, hopefully, Lode having a possibility of performing experiments with a fixed value of μ parameter ($\mu=-1,-0.5,0,0.97$) was able to made the draw a curve $f_{exp}=(\sigma_1-\sigma_3)/\sigma_z$ where σ_z measured normal stress which has a meaning the equivalent stress (Fig. 2).

The Lode diagram has a great generality – it is enough to replace σ_2 by μ and to reorganize any criterion to a requested form $(\sigma_1-\sigma_3)/\sigma_{eq}=f(\mu)$ (see Fig. 2).

Lode, having experimental curve, was able to compare it with some theoretical curves based on Tresca, Beltrami, Huber hypothesis as (Fig. 3)

$$\begin{aligned}
f_{Tresca}&=\frac{\sigma_1-\sigma_3}{\sigma_{Tresca}}=1=\text{const}\\
f_{Beltrami}&=\frac{\sigma_1-\sigma_3}{\sigma_{Beltrami}}=\sqrt{\frac{1}{1+\frac{1}{4}(1+\mu)^2-\nu(1+\mu)}}\\
f_{HMH}&=\frac{\sigma_1-\sigma_3}{\sigma_{HMH}}=\frac{\sigma_1-\sigma_3}{\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1}}=\frac{2}{\sqrt{3+\mu^2}}
\end{aligned} \tag{1.5}$$

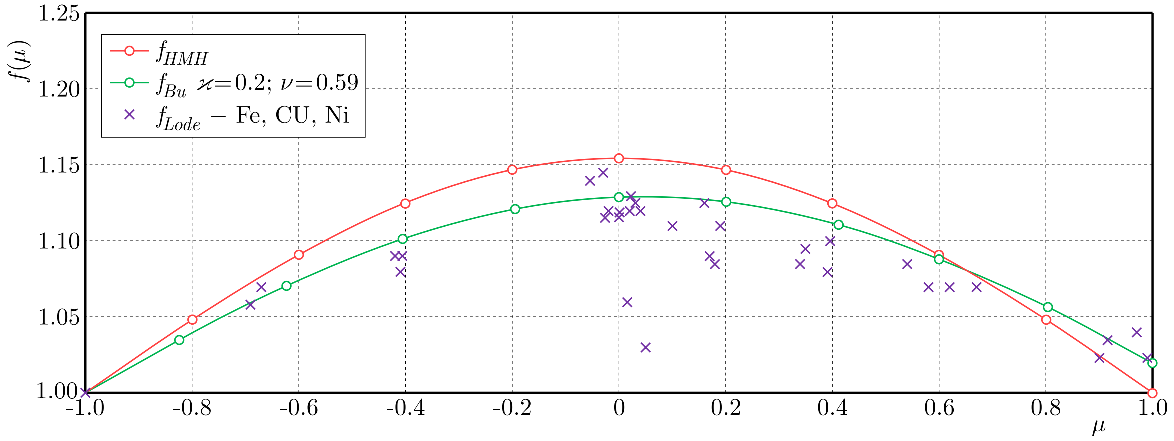

Fig. 2. The Lode diagram showing the influence of stress shearness into a state of material effort. A dimensionless pure shear factor μ, Eq. (1.4), gives a possibility to compare different materials like iron, cooper, nickel. In above, $f_{HMH} = (\sigma_1 - \sigma_3)/\sigma_{HMH} = 2/\sqrt{3+\mu^2}$ and $f_{Bu} = (\sigma_1 - \sigma_3)/\sigma_{Bu}$, where σ_{Bu} is determined by Eq. (1.7)

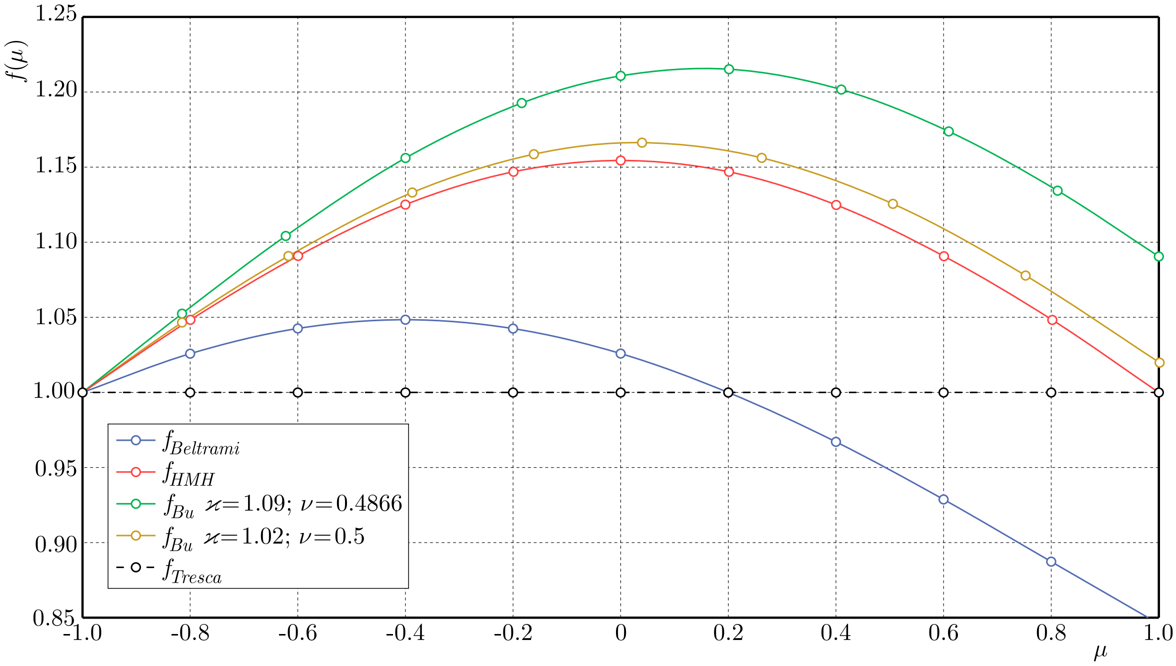

Fig. 3. The Lode diagram showing comparison of theoretical curves: Tresca's, Beltrami's (with Poisson's coefficient $\nu = 0.3$), Huber's, and two Burzyński's

Nowadays, one can add the Burzyński curve

$$f_{Bu} = \frac{\sigma_1 - \sigma_3}{\sigma_{Bu}} \tag{1.6}$$

With the equivalent stress (Dudda, 2021)

$$\sigma_{Bu} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{2\varkappa}\left\{(\varkappa - 1) + \sqrt{(\varkappa - 1)^2 + 4\varkappa\Big[(1+\overline{\nu})\frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{(\sigma_1 + \sigma_2 + \sigma_3)^2} - \overline{\nu}\Big]}\right\} \tag{1.7}$$

where the Burzyński coefficients are $\varkappa = k_c/k_t$, $\overline{\nu} = k_c k_t/2k_s^2 - 1$.

From Fig. 2, it follows that the Huber hypothesis is better fitting with the experimental data then the Tresca curve (compare Figs. 3 and 2). The Beltrami hypothesis is always wrong. The

Burzyński curves can be calibrated using appropriate values of $\varkappa$ and $\overline{\nu}$. In Burzyński's curve we have the boundary condition $f_{Bu} = \varkappa$ for $\mu = 1$.

Now, it is interesting to compare the Huber and Mohr hypotheses. The task is to demonstrate that the equivalent Huber stress calculated according to the deformation energy is always smaller than the equivalent stress calculated according to the Tresca hypothesis of the greatest shear stresses. To do this, it is necessary to separate from the definition of σ_{HMH}, before the root, the term $\sigma_1 - \sigma_3 \equiv \sigma_{Tresca}$

$$\sigma_{HMH} = \frac{1}{\sqrt{2}}(\sigma_1 - \sigma_3)\sqrt{\Big(\frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3}\Big)^2 + \Big(\frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}\Big)^2 + 1} \tag{1.8}$$

And further, eliminating, as Burzyński does, the mean stress with the help of the Lode parameter μ

$$\sigma_2 = \frac{1+\mu}{2}\sigma_1 + \frac{1-\mu}{2}\sigma_3 \tag{1.9}$$

We obtain

$$\sigma_{HMH} = \sigma_{Tresca}\sqrt{\frac{3+\mu^2}{4}} \tag{1.10}$$

from which one can see that since $-1 < \mu < 1$ is even for $\mu = 0$, $\sigma_{HMH}/\sigma_{Tresca} = 0.866$. It means, in practice that the HMH hypothesis is safer. But for materials with $\varkappa \neq 1$ the Burzyński is safer then HMH. For instance, for pure compression $\mu = 1$, we obtain $f_{Bu} = \varkappa$, which means that $\varkappa$ changes the shape of limiting curve.

The above operation one can found in the monograph of Krzyś and Życzkowski (1962). This result does not surprise Krzyś and Życzkowski, as they add that "the correctness of the statement is immediately visible". Therefore, safer hypotheses should be used. What is a valuable didactic element of Krzyś and Życzkowski's book is task number 14.7, in which the equivalent stresses calculated from various hypotheses are compared. Let us also recall the outstanding students of the Cracow School of the Material Effort. These are, among others: Jacek Skrzypek, Artur Ganaczarski, Halina Egner, Błażej Skoczeń, Kinga Nalepka. It is worth remembering that the geometric method of Mohr's envelope is a part of the very classical approach, yet developed in Hellenic and Medieval Mechanics, which mechanics, due to their tools, were called as "geometric mechanics". Also, the whole of Newtonian mechanics was expressed using geometric tools and descriptive constructions. It was not until Euler and Lagrange that analytical mechanics were introduced. Therefore, the dispute between Mohr and Huber was de facto a dispute over the superiority of "geometric mechanics" over "analytical mechanics". One of the formal obstacles to the construction of Mohr's circle is the lack of algebraic theorems corresponding to its graphic constructions.

We should remember Burzyński's question "Are Mohr's envelopes the full content of his effort hypothesis, or are they illustrations of some unknown hypothesis, which is a mere means of drawing, more or less successful?" (Burzyński, 1929a). Or speaking in a more contemporary language: which scientific tool energetic or geometrical is more correct.

2. The Tresca-Mohr geometrical approach – a critical review

2.1. Ultimate stresses approach

Let us note that the geometrical approach of Otto Mohr was based on "descriptive geometry techniques". Mohr used his representation of stresses σ_1, σ_2, σ_3 by circles to devise the material effort hypothesis which can be adopted to various stress conditions. He assumed that it was the maximum shearing stress (Mohr, 1906). Under such circumstances, it is necessary to consider only the largest circle. Mohr called it the principal circle and suggested that such circles should be constructed when experimenting for each stress condition in which the failure occurred. In Fig. 4, the cast iron is tested to fracture in tension I, compression II and pure shear III, and three principal circles I, II, III are depicted. If there is a sufficient number of such principal circles, an envelope of these circles can be drawn. It can be assumed that for any stress condition for which there is experimental data, the corresponding limiting principal circle will also touch the envelope.

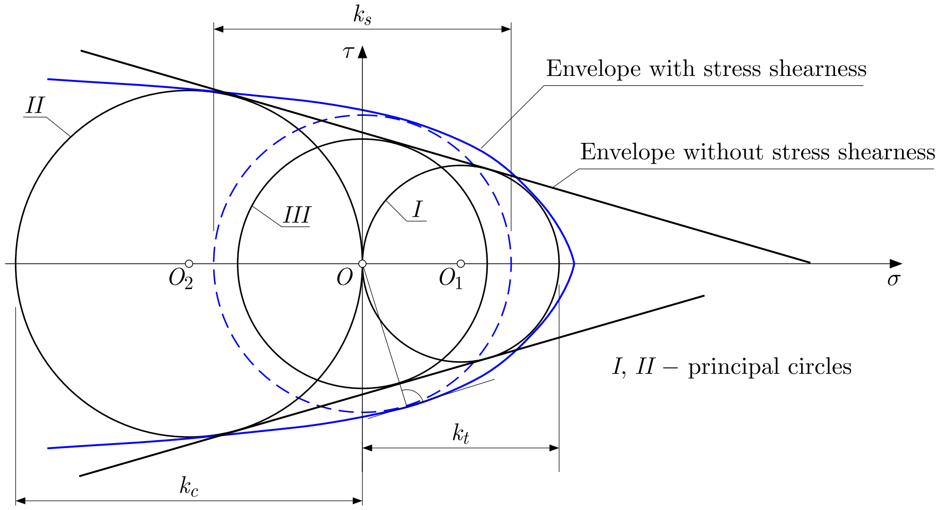

Fig. 4. Mohr limiting the yield surface as an envelope of principal circles: circle I with the radius k_t corresponds to uniaxial tension; circle II with the radius k_c corresponds to uniaxial compression; circle III with the radius k_s corresponds to pure shear. For determination of Mohr's plane, any unit vector $\mathbf{n} \cdot \mathbf{n} = 1$ must be taken, and two Mohr invariants are determined as follows: $\sigma = \sigma_{ij} n^i n^i$, $\tau^2 = \sigma_{ik}\sigma_{kj} n^i n^j - \sigma^2$

Note that one needs three cycles for a whole envelope. For example, when considering the cast iron, Mohr suggested that an envelope be taken as the two outer tangents to circles I and II. The limit strength in shear k_s is then found by drawing circle III, which has its center at O and is tangential to the envelope. It means, if k_t and k_c are values of ultimate strength in tension and compression, one can find from Fig. 4. that the ultimate strength in shear is $k_s = k_t k_c/(k_t + k_c)$, which agrees satisfactorily with the conical Drucker-Prager hypothesis (Drucker and Prager, 1948). In other case (broken line), when k_s is much greater than the arithmetic mean, the envelope must be constructed as an elliptical or paraboloidal section of a yield surface. For instance, in a case of St12T steel measured at temperature $T = 20°\mathrm{C}$, $k_s = 340\,\mathrm{MPa}$ (Dudda, 2020) but from $k_s = k_t k_c/(k_t + k_c)$ it follows that $k_s = 290\,\mathrm{MPa}$ – which means that the envelope cannot be a straight line. It is known from the literature (Kolupaev *et al.*, 2016; Nowak *et al.*, 2014; Olifieruk *et al.*, 2004; Skrzypek and Ganczarski, 2016; Dubey *et al.*, 2023) that measurement of k_s with appropriate accuracy is a difficult task. In needs realization of the state of pure shear where the principal stresses are $\sigma_3 = -\sigma_1$ and $\sigma_2 = 0$. Especially, making $\sigma_2 = 0$ is difficult. It means that a state of "stress shearness" is responsible for partial contribution of σ_2 into the material effort concept.

2.2. Geometrization by von Mises

The first continuation of geometrical approach was October 1913, when Richard von Mises (1913) wrote his paper on the foundations of modelling of plastic bodies. In that time, three-

-dimensional continuum mechanics was already a well-established field of knowledge, but Mises nevertheless decided to derive mathematical models "from the very beginning" in it, resulting in a fairly complete framing of the mechanics of plasticity within the framework of a new "plastic flow model" – today called the Mises flow model.

The question of "material effort" (German: die Anstrengungshypothesen) does not appear in von Mises's work, and only the "condition of plasticity" is discussed. At the beginning, Mises showed how to construct octahedral invariants of the stress tensor, which, following Mohr, are most often built in the principal axes, looking for the smallest and largest shear stress – Mises denotes them by τ_1, τ_2, τ_3, and defines accordingly

$$2\tau_1 = \sigma_2 - \sigma_3 \qquad 2\tau_2 = \sigma_3 - \sigma_1 \qquad 2\tau_3 = \sigma_1 - \sigma_2 \tag{2.1}$$

Further, von Mises emphasizes that an important invariant is the sum of squares of the octahedral stresses $\tau_1^2 + \tau_2^2 + \tau_3^2$, which today is nothing more than $0.5\sigma_{HMH}^2$. A key to Mises's work, the equation is expressed as

$$\tau_1^2 + \tau_2^2 + \tau_3^2 = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{2}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \tag{2.2}$$

which nowadays can be written as

$$\tau_1^2 + \tau_2^2 + \tau_3^2 = \frac{1}{2}\sigma_{HMH}^2 \tag{2.3}$$

It is an accidental result when both geometrical and energetic approaches lead to the same formulae. Next, von Mises raises the question of experimental motivations. He found an assumption that in the system of principal stresses, the sum of octahedral stresses at the point at which the elastic limit is reached is equal to zero

$$\tau_1 + \tau_2 + \tau_3 = 0 \tag{2.4}$$

Commenting on this condition, von Mises thanks Mohr, who was the first to analyse this condition at the time of reaching the elastic limit k_s

$$|\tau_1| \leqslant k_s \qquad |\tau_2| \leqslant k_s \qquad |\tau_3| \leqslant k_s \tag{2.5}$$

Von Mises showed it in the figure – in the axes of octahedral stress (Fig. 5), as a cube – when one cuts this cube with a plane, one gets a parallel hexagon in the cross-section.

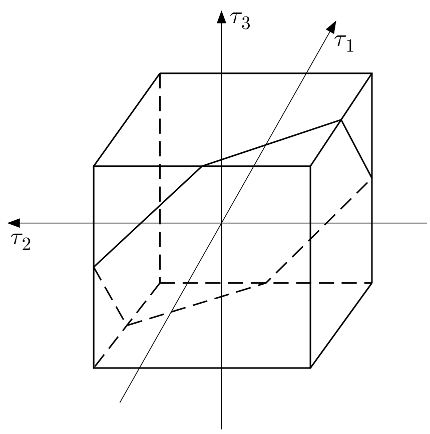

Fig. 5. Mohr's boundary surface in the form of an octahedral cube – interpretation of von Mises (1913, Fig. 4)

Then von Mises, in complete isolation from the energy methods and under the influence of Mohr's geometricism, proposed to modify the surfaces of elastic states in such a way as to break away from Mohr's previous condition and allow only the vertices of this diagram to come into play. In this way, it will be possible to bypass the assumption of invalidity of the central principal stress and, in addition, it will allow one to replace the diagram with a single simple solid without edges, e.g. of the "circumscribed circle" type. Thus, in place of the cube (Eq. (2.5)), von Mises proposed a cylinder as

$$\tau_1^2+\tau_2^2+\tau_3^2=2k_s^2 \qquad \text{or} \qquad \frac{1}{2}\sigma_{HMH}^2=2k_s^2 \qquad \text{or} \qquad \sigma_{HMH}^2=4k_s^2 \tag{2.6}$$

That is, taking into account that $k_s = k_r/2$, we have contemporary

$$\sigma_{HMH}^2-k_t^2=0 \tag{2.7}$$

And he added, "it is obvious that this condition is much simpler to describe analytically, because there are no ambiguities in the corners". In that time corners led to mathematical problems with which Mises's good friend [later wife] Hilda Geiringer had so much trouble (Geiringer and Prager, 1934).

Von Mises ignored the fact that condition (2.6) is physically "wider" (more capacious) than Mohr's condition (2.5). Let us emphasize that the spirit of this physics-free geometrization of Mohr's condition is still prevalent in the literature today and is even highly respected. Despite its apparent attractiveness, this approach should be considered a "wrong direction of research", leading astray determined by modern geometricism, which has dominated mechanics for years.

A generalization of Mises's geometrical reasoning assumes that one can raise the corner condition to any power, and then one has

$$|\sigma_1-\sigma_2|^m+|\sigma_2-\sigma_3|^m+|\sigma_3-\sigma_1|^m=mk_s^m \tag{2.8}$$

when $m = 1$ and $m \to \infty$ we have the Tresca condition, $m = 2$ the von Mises condition, at $m < 1$ we have a concave boundary surface. The cross-sections in the deviator plane are shown in Fig. 6.

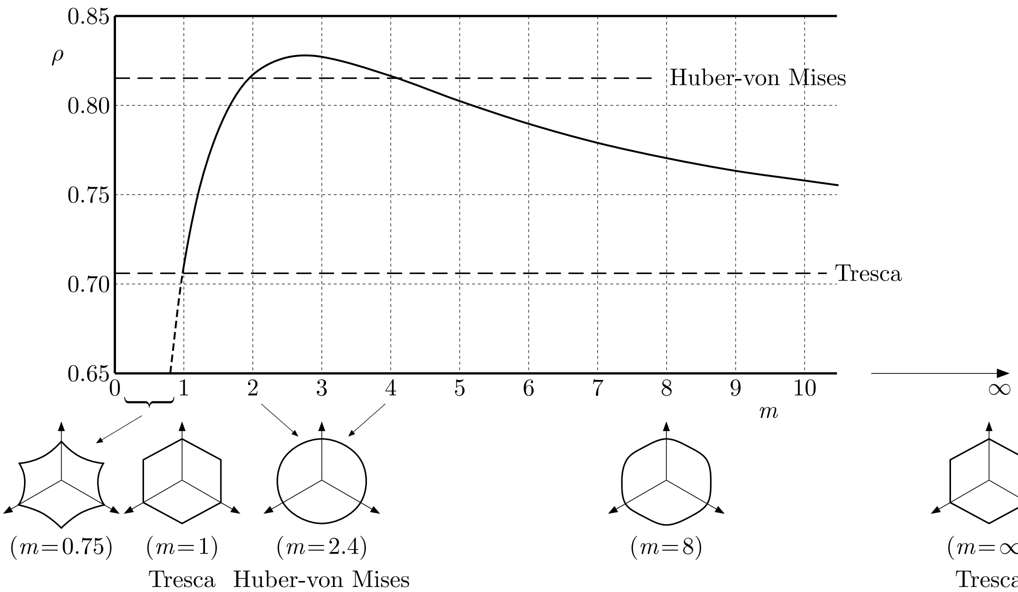

Fig. 6. Drawings of a generalized von Mises surface (Skrzypek, Ganczarski, 2015, p. 165), here $\rho = \sqrt{2J_2}$ is a cylindrical coordinate in the Haigh–Westergaard diagram that differs from the intensity $\sigma_i = \sqrt{3J_2}$

It is worth mentioning that the effect of partial appearance σ_2 within a yield function was experimentally discovered by Lode (1926) in combined tension/compression and internal pressure tests, which was a modernization that consequently omitting the role of σ_2 and was still consistent with the Mohr envelope concept. The second example was the Drucker two-parameter modernization of Mohr's condition, Drucker (1949)

$$\sqrt{J_2^3} - cJ_3 - 3k_s^3 = 0 \tag{2.9}$$

where J_3 changes the role of equal weight for σ_1, σ_2, σ_3 condition and provides a partial contribution of σ_2. Both models Eq. (2.8) and Eq. (2.9) have two unknown constants m, k_s and c, k_s, respectively. They can be expressed on the base of the same experimental data k_c and k_t. For instance, Hosford and Allen (1973) found that for bbc crystals the best fitted calibration was $m = 1.6$ and Cazacu and Barlat (2004) for bbc polycrystals calculated that for $k_c/k_t = 1/(1.28)$ the coefficient c in Eq. (2.9) was equal to 0.92.

2.3. The geometrical stress shearness

Having defined the stress triaxiality (normalized first invariant I_1), the stress intensity (von Mises norm of the second invariant J_2) is (Mises,1913)

$$\eta = \frac{\sigma_m}{\sigma_{HMH}} = \frac{I_1}{3\sqrt{3J_2}} \qquad \sigma_i = \sqrt{2J_2} = |\boldsymbol{\sigma}| \tag{2.10}$$

Now let us discuss the stress shearness concept treated to be a normalized third invariant of the deviatoric stress tensor J_3. The stress shearness is related with the old question discussed between German scientist concerning the appearance (or not) of the second principal stress σ_2 within some material effort hypothesis based on Tresca, Coulomb and, first of all, Mohr concepts of the "maximum shearness". It was shortly after The First World War, and after Guest work (Gao *et al.* 2011), when in many experiments, the question of priority between the two concepts: the maximum shearness and energy-based hypothesis, were researched and determined.

The general conclusion, achieved from Lode's experiments, was that the definition of the state of material effort as well as definition of yield surface are σ_2 dependent. It means, that σ_2 cannot be omitted, like in the Mohr-Coulomb criterion, or cannot be completely taken as in the energy-based Huber-Mises, but contribution of σ_2 should be made in an intermediate manner. It can be geometrically represented within the stress space when it is parametrized by Heigh-Westergaard cylindrical coordinates h, r, θ, where $h = I_1/\sqrt{3}$, $r = \sqrt{2J_2}$, and θ is the Lode angle within the range $0 \leqslant \theta \leqslant \pi/3$. The Lode angle is related with the normalized third invariant ξ (Bai and Wierzbicki, 2010) as

$$\xi = \cos(3\theta) = \cos\frac{\pi(1-(\theta)}{2} = \frac{3\sqrt{3}}{2}\frac{J_3}{\sqrt{J_2^3}} \tag{2.11}$$

determined in the range $-1 \leqslant \xi \leqslant 1$ with values $\xi = 0$ at pure shear $\theta = 30°$, $\xi = 1$ at uniaxial tension $\theta = 0°$, and $\xi = -1$ at uniaxial tension $\theta = 60°$. It works with the Lode parameter (Lou *et al.*, 2014)

$$\mu = \frac{3\tan(\theta) - \sqrt{3}}{\tan(\theta) + \sqrt{3}} \tag{2.12}$$

and with the normalized Lode angle

$$\overline{\theta} = 1 - \frac{6}{\pi}\theta = 1 - \frac{2}{\pi}\arccos\xi \tag{2.13}$$

within the range $-1 \leqslant \overline{\theta} \leqslant 1$.

In general, a state of material effort should be a function of three parameters: stress triaxiality, stress intensity and stress shearness. The same concerns the limit surface – being for instance a yield surface, rupture surface, strength surface, etc. Any constructed yield surface can be defined in the Bai-Wierzbicki space spanned on (η, σ_i, ξ) or $(\eta, \sigma_{HMH}, \xi)$ parameters – such an approach is developed in the papers by Yoon *et al.* (2014), Lou *et al.* (2020), Wierzbicki *et al.* (2005).

2.4. The Lode angle corrections

Generally, the limit surface, especially for pressure sensitive materials, are developed in the three-dimensional space of main stresses σ_1, σ_2, σ_3 in the co-called Heigh-Westergaard cylindrical coordinates (h, r, θ). In this approach, as experimental results show, the important is shape of the deviatoric cross-section, which is perpendicular to the hydrostatic axis (Prager and Hodge, 1954).

This fact led to the proposal of many different shape functions $\mathcal{P}(\theta)$ that are only dependent upon the Lode angle $r = r_0\mathcal{P}(\theta)$ (Podgórski, 1984, 1985) where the value of $r_0 = r(\theta = 0)$. For instance, the most known Gudehus surface for pressure-insensitive materials (Gudehus, 1973), the shape function $\mathcal{P}(\theta)$ takes a very simple form $r^2 = r_0^2(1-\xi)$, where $\xi = \cos(3\theta)$. In the Lade and Duncan (1973) paper, the yield surface is based on one parameter shape function

$$\mathcal{P}(\theta) = \Big[\cos\Big(\frac{1}{3}\arccos\alpha\xi\Big)\Big]^{-1} \tag{2.14}$$

where $\alpha = \text{const}$, satisfying the condition $0 \leqslant \alpha \leqslant 1$. In such an approach, the well-known internal friction angle ϕ, appearing in the Coulomb-Mohr model $|\tau_n| + \sigma_n \tan\phi = c$, can be easily incorporated

$$\mathcal{P}(\theta) = \Big[\cos\Big(\frac{1}{3}\arccos\xi - \phi\Big)\Big]^{-1} \tag{2.15}$$

It means that, in general, the shape function can be dependent on two arbitrary constantans

$$\mathcal{P}(\theta) = \cos(30-\phi)\Big[\cos\Big(\frac{1}{3}\arccos\alpha\xi - \phi\Big)\Big]^{-1} \tag{2.16}$$

which is a generalization of research for large class of materials including metals, rocks, concrete and soils (Lade and Duncan, 1973; Willam and Warnke, 1974; Podgórski, 1985). Here, the angle ϕ has a simple physical interpretation of an internal friction angle.

2.5. Strength differential effect

Modelling of the material effort that takes into account the strength differential effect was a subject matter of many pioneers and veterans like Coulomb, Tresca, Lame, Clapeyron, de St. Venant, Navier. The first satisfying modification of the limit stresses hypotheses was proposed in 1856 year by R.W. Rankine as (Yu, 2004)

$$\sigma_1 \leqslant k_t \qquad \text{and} \qquad \sigma_3 \leqslant -k_c \tag{2.17}$$

where σ_1, σ_2, σ_3 are three principal values of the stress tensor – ordered from tension to compression. If $\varkappa = k_c/k_t = 1$ then this effort hypothesis reduces to the historical Galileo one. In the Heigh-Westergaard principal stress picture, the Rankine hypothesis is represented by six plains, double parallel, that build a cube with size $k_c + k_t$ located parallely to main axes – this model is pressure insensitive, of course.

Another hypothesis for the strength differential, but pressure insensitive material was thought as an extended Coulomb-Tresca by Otto Mohr in 1906 year (Mohr, 1906). He used his "geometrical approach" known as "curve envelope" (germ. Anstalung kurve) where the material in a limit state undergoes slip due to maximal shear and, additionally, small cleavage due to presence of "longitudal stress" governed by $k_c - k_r$ difference. Analytically, the Mohr strength differential hypothesis reads

$$(\sigma_1 - \sigma_3)^2 + (k_s - k_t)(\sigma_1 + \sigma_3) = k_c k_t \tag{2.18}$$

In the Haigh-Westergaard principal stress picture this function forms a paraboloid moved to infinity, which possesses symmetry in the pressure axes. The mximum positive (tension) stresses in the Mohr paraboloid, located in the pressure axes, is obtained from the envelope with radius $a = 0.25(k_c - k_r)$ – in the case of $k_c = k_r$, the hypothesis turning into the classical Tresca-Guest one $\tau_{II} = 0.5(\sigma_1 - \sigma_3) = k_s$. Let us mention that the shear limit k_s is defined from the beginning as $k_s = 0.5k$ where $k = k_c = k_t$. The form of equation (2.18) implies that the limit on the pure shear is now defined to be

$$k_s = \frac{1}{2}\sqrt{k_c k_t} \tag{2.19}$$

It means that having measured three limits k_c, k_t, k_s and Mohr's relation (2.19), one can check correctives of hypothesis (2.18). Let us note, that Mohr's hypothesis (2.5) belonging to the so-called shearing-dominated mode of limit behaviour was also developed by Coulomb, Guest and others (Życzkowski, 1999; Hu *et al.*, 2017). This approach prefers octahedral state of stresses and τ_I, τ_{II}, τ_{III} shearing stresses that naturally leads to a non-smooth yield surface with edges. The edges are simple results of combination of non-continuous yield conditions, and the number of edges is a simple consequence of the number of field function. The fundamental are the six Tresca conditions

$$\tau_I \leqslant \mp k_s \qquad \tau_{II} \leqslant \mp k_s \qquad \tau_{III} \leqslant \mp k_s \tag{2.20}$$

written for no ordered main stresses σ_1, σ_2, σ_3. In this case, the number of edges (or pieces of limit surface) is six ($m = 6$). Similarly, for the Rankine hypothesis there is $m = 6$. But in the Galileo, Mariotte, and Ivliev hypothesis one has only three edges ($m = 3$), the Kolupaev hypothesis nine edges ($m = 9$), the Sokolovsky – eighteen edges ($m = 18$) (Kolupaev *et al.*, 2016).

2.6. Pressure sensitiveness within the maximum shear stresses treatment

A similar conclusion concerning the two-parameter shape function can be obtained by a direct analysis of the yield surfaces based examples on J_3 invariant, for instance, for pressure-insensitive $I_1^3/J_3 = k$ (Lade and Duncan, 1973); $J_2^3 + cJ_3^2 - k^6 = 0$ (Drucker, 1949); $J_2^{3/2} - cJ_3 - k^3 = 0$ (Cazacu and Barlat, 2004) as well as for pressure sensitive materials $a(bI_1^6 + 27J_2^3 + cJ_3^2)^{1/6} = 0$ (Gao *et al.*, 2011); a$[bI_1 + (J_2^{3/2} - cJ_3)^{1/3}] = 0$ (Yoon *et al.*, 2014) and $aI_1^2 + bJ_2 + cJ_3^{2/3} - 1 = 0$. Going to show this, let note that for isotropic, pressure-insensitive materials the above limit conditions can be written in terms of the second and third invariants as

$$f(J_2, J_3, k, c, n) = \sqrt{J_2^{3n}} - cJ_3^n - k^{3n} = 0 \tag{2.21}$$

There are only three parameter (k, c, n) yield surfaces – for $n = 1$, one obtain the Cazacu and Barlat (2004) condition, however, for $n = 2$ we have the condition of Drucker (Drucker,1973), yet a more simple example $J_2^3 - k^6 = 0$ was considered by Reuss (1928). Let us observe that for even exponents $n = 2, 4, 6, \ldots$ the yield condition predicts the same values of the yield stress in compression and tension $k_c = k_t$. Differently, for odd exponents $n = 1, 3, 5, \ldots$ the yield condition provides different values of the yield stress, exhibiting the SD-effect. Next, reorganizing Eq. (2.21), one obtains

$$f(J_2,J_3,k,c,n)=\sqrt{J_2^{3n}}\Big[1-c\Big(\frac{J_3}{\sqrt{J_2^3}}\Big)^n\Big]-k^{3n}=\Big(\frac{1}{2}r\Big)^{3n}\mathcal{P}'(\theta)-k^{3n}=0 \tag{2.22}$$

where, now, the generalized shape function $\mathcal{P}'(\theta)=1-c_1\xi^{3n}$, $c_1=c(2\sqrt{3}/9)^n$. It is easy to observe that condition, Eq. (2.22), has the same physical meaning as $f=r-k=r_0\mathcal{P}(\theta)-k=0$ in the classical Coulomb-Mohr-like conditions, but the shape function $\mathcal{P}'(\theta)$ is only one-parametrical, whereas $\mathcal{P}(\theta)$ is two-parametrical.

There is in the literature unclearness concerning pressure sensitive materials. Namely, some authors, like Mirone and Corallo (2010), take into account the expression I_1J_2/J_3, which cannot lead to simple splitting of distortional and volumetric effects. But in the case when such splitting can be postulated, a yield function, written within the cylindrical (h,r,θ) coordinates has a form

$$f(I_1,J_2,J_3)=\Big(\frac{1}{2}r\Big)^{3n}\mathcal{P}'(\theta)-k_1h\mathcal{P}''(\eta)-k_2^{3n}=0 \tag{2.23}$$

Here $\mathcal{P}'(\theta)$ is interpreted as the circumferential shape function and $\mathcal{P}''(\eta)$ is longitudinal along the hydrostatic axis $\sigma_1=\sigma_2=\sigma_2$ shape function, depending on the stress triaxiality $\eta=\sigma_m/\sigma_{HMH}$ that change with $h=I_1/\sqrt{3}$. In the simplest form this condition is a three-parametrical one. However, when the invariant J_2 is removed from considerations – like in $I_1^3/J_3-c=0$ – the above form should be modified to:

$$f(I_1,J_3)=h\mathcal{P}''(\eta)\mathcal{P}'''(\theta)-k=0 \tag{2.24}$$

where both shape functions are multiplied. The similar form of the yield surfaces for a pressure sensitive material has been proposed by Barlat *et al.* (2003). In general, such a yield function depends only on I_1 and $J_2/\rho(\theta)$, where $\rho(\theta)\mathcal{P}^{-1}$ is the circumferential shape function – the deviatoric sections in planes with constant I_1 (or h) look similar but not necessarily coincide. The deviatoric sections are circular when the shape function does not depend on the Lode angle. Then $\rho(\theta)=1$. Since θ varies between $-\pi/6$ and $\pi/6$ a shape function should contain periodic curves with the period $2\pi/3$. Some application of the $\rho(\theta)$ shape function has been applied to bone fracture mechanics by Pietruszczak *et al.* (1999) in the form

$$f=a_1\frac{\sqrt{3J_2}}{\rho(\theta)k_c}+a_2\frac{3J_2}{(\rho(\theta)k_c)^2}-\Big(a_3+\frac{I_1}{k_c}\Big)=0 \tag{2.25}$$

where a_1, a_2, a_3 are dimensionless constants and k_c is the yield strength in compression. This model can be prolonged onto anisotropic geomaterials (Piertuszczak and Mróz, 2001).

Now the examples of pressure sensitive materials like pressure modified Tresca

$$|\tau_n|=c-\frac{1}{3}\epsilon(\sigma_1+\sigma_2+\sigma_3) \tag{2.26}$$

or pressure modified Coulomb-Mohr

$$|\tau_n|+\sigma_n\tan\phi=c-\frac{1}{2}\epsilon(\sigma_1+\sigma_2+\sigma_3) \tag{2.27}$$

can be reformulated in terms of the shape function $\rho(\theta)\equiv\mathcal{P}^{-1}$. For instance, the pressure modified Coulomb-Mohr yield surface takes the following form (Barlat *et al.*, 1991)

$$\sqrt{J_2}(3\cos\theta-\sqrt{3}\sin\phi\sin\theta-\epsilon I_1\sin\phi-3c\cos\phi=0 \tag{2.28}$$

where ϵ is the pressure coefficient, ϕ is the friction angle and c is cohesion.

It has, generally, been assumed that the state of the maximum stress shearness does not need appearance of σ_2 in the mechanical or thermal effort modeling. Numerous continua, like dry sand, obeys the Tresca or Coulomb-Mohr conditions, in which failure does not need any previous plastic or inelastic deformation. When the stress shearness dominates over the stress intensity or stress triaxiality, the slip in the σ_1-σ_3 plane is independent of σ_2. Bur in many crystalline metals, when plastic flow occurs firstly, motion of dislocations on available slip planes within each individual crystal possesses at least five independent slip systems which are required in each grain if it is to undergo shape changes imposed by the crystals around it (Pęcherski, 1998). This means that even if the macroscopic plastic deformation is apparently confined in σ_1-σ_3 plane within a high proportion of individual grains, the plastic flow must be occurring on slip planes inclined to this, i.e. in slip systems driven by the intermediate main stress σ_2. It is therefore, physically reasonable that all three principal stresses should appear in form of a limit surface: proportional, yield, strength, fracture, and so on. But between the full appearance of σ_2 and the omitting of σ_2 – there is an open domain with a different level of the stress shearness which is represented via the Lode parameter. In another words, the passage from null-dependence to full-dependence of σ_2 within limit surface formulae can be realized continuously by using the Lode parameter μ (or the Lode angle θ, the normalized Lode angle $\overline{\theta}$, the normalized third invariant ξ).

2.7. An extension of the Mohr geometrical treatment of anisotropic materials

The Tresca and Coulomb-Mohr concept of maximum shear stresses can be applied also to anisotropic materials like wood, stones, clays and polycrystalline metals. To describe plastic anisotropy of rolled metal sheets, Hill (1948) developed an extension of the Mises-Hosford isotropic condition, Eq. (2.2), to the anisotropy case – this solution was based on 6 unknown coefficients. The rolling direction, the long transverse direction and short transverse direction are three main directions where the unknown parameter should be calibrated by the experimental data – generally one can obtain three-time more data than for isotropic bodies. Up to 1948 year, several methods had been proposed (Baltov and Sawczuk, 1965; Życzkowski, 1981; Mróz, 1967). A review of different solutions can be found in the paper by Oana Cazacu and Barlat (2004). They proposed, finally, to explore the technique of linear transformations $\sigma'_{ij} = L_{ijkl}\sigma_{kl}$ and $\sigma''_{ij} = C_{ijkl}\sigma_{kl}$ to define an anisotropic yield surface

$$|\sigma'_1 - \sigma''_2|^m + |\sigma'_2 - \sigma''_3|^m + |\sigma'_3 - \sigma''_1|^m = k_s^m \tag{2.29}$$

that needs fourteen parameters to be calibrated (six L_{ijkl}, six C_{ijkl} and m, k_s). For a pressure sensitive material the above criterion was extended by (Kuroda and Kuwabara, 2002) as

$$(1-c)\big(|\sigma'_1 - \sigma''_2|^m + |\sigma'_2 - \sigma''_3|^m + |\sigma'_3 - \sigma''_1|^m\big) + c\rho\big(|\sigma_1|^m + |\sigma_2|^m + |\sigma_3|^m\big) = k_s^m \tag{2.30}$$

where one additional constant c appears, and $\rho = 3^m(2^{m-1}+1)^{-1}$. The whole family of different dedicated yield surfaces is coming from the above condition, but calibration of such great numbers of parameters needs many experimental data connected with crystal plasticity calculations (Hu and Yoon, 2021).

From another point of view, the maximum shear approach cannot take into account the Lodge parameter or Lode angle directly. Thus, a way by using J_3 invariant is more promising. Cazacu and Barlat (2004) proposed a manner of introducing anisotropy into J_2 (6 parameters) and J_3 (11 parameters) invariants. A quite similar approach was developed by Yoon *et al.* (2014) where the concept of linear transformation was applied to all three invariants independently. It is the first invariant I_1*, second invariant J'_2 and third invariant J''_3, which are based on the transformed Cauchy stress tensor $\boldsymbol{\sigma}^* = \mathbf{H}\boldsymbol{\sigma} = h_1\sigma_1\mathbf{e}_1\otimes\mathbf{e}_1 + h_2\sigma_2\mathbf{e}_2\otimes\mathbf{e}_2 + h_3\sigma_3\mathbf{e}_3\otimes\mathbf{e}_3$, $\boldsymbol{\sigma}' = \mathbf{L}'\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'' = \mathbf{L}''\boldsymbol{\sigma}$ and its deviator. The Yoon model needs to calibrate $3+6+6 = 15$ unknown parameters of $\mathbf{H}$,

$\mathbf{L}'$, $\mathbf{L}''$. Among these anisotropic parameters, eight of them are related with in-plane properties while the other four parameters are used to describe the through-thickness behaviour of the metal. Finally, the yield function for pressure sensitive anisotropic metals reads in the form of (Yoon *et al.*, 2014)

$$I_1^* + \sqrt[3]{\sqrt{J_2'^3} - J_3''} = 1 \tag{2.31}$$

This yield function is able to describe: the stress triaxiality, the stress intensity and the stress shearness (Lode parameter) effects within the scope of different anisotropy of metals (Nixon *et al.*, 2010).

3. Energy-based limit approach

3.1. Pioneering steps – Maxwell, Beltrami, Huber, Schleicher, Burzyński

Recall, that the "father" of energy-based approach is James Clerk Maxwell who in 1856, in the letter to William Thomson, without introducing a concept of "distortional energy" as a part of elastic strain energy, wrote that it was the best candidate to be a measure of material effort (Maxwell, 1856; von Helmholtz, 1903; Pęcherski, 2008; Altenbach, 2010). Modern reconstruction of Maxwell's energy-based approach was worked out by Rychlewski (2011).

In the complete mathematical form, this energy-based approach was initiated by Beltrami (1885) in the form $W = \Phi \leqslant K$, where $\Phi = [(1-2\nu)/3E]I_1^2 + [(1+\nu)/3E]3J_2$ is some elastic deformation energy (volumetric density) describing a state of material effort, and K is critical value of this energy. Beltrami firstly found how critical energy K depended on the uniaxial yield k_t or torsion k_s (Becchi, 1994). This approach also opens a possibility of using many other experimental data like Vickers hardness, Charpy critical energy (fracture toughness), cohesiveness critical energy, and so on (Orłowski *et al.*, 2020, 2013).

Explicitly, the concept of "specific work of strain" Φ (internal energy, or specific work of stress) as a measure of material effort (germ. *Die Anstrengung*) was developed by Huber (1904), who was able to introduce a notion of "equivalent stress" (or reduced stress σ_{eq}). Also, he proposed the first picture of a limiting surface within the space of three principal stresses (Huber, 1904, his Fig. 2). Next, by comparison of the Beltrami and de Saint Venant criterions, Huber performed an exhausting discussion on the role of the Poisson ratio coefficient and its appearance in energy-based hypothesis. Next, he found that the Maxwell-Helmholtz decomposition of strain energy into purely volumetric and distortional parts $\Phi = \Phi_\nu + \Phi_f = [(1-2\nu)/3E]I_1^2 + [(1+\nu)/3E]3J_2$ could be useful in further research, and discussed a case when volumetric energy Φ_ν was negligible. Finally, Huber proposed a new combined criterion: if $p \geqslant 0$ then $\Phi \leqslant K$ and $p \leqslant 0$ then $\Phi_f \leqslant K_f$ – this condition leads to an elliptical-cylindrical yield surface and nowadays is called "the Beltrami-Huber combined condition".

3.2. Combined Huber's hypothesis

But presently and literally, when we go into the details of this reasoning, we have three hypotheses to offer: Beltrami's (ellipsoid), Huber's (part ellipsoid, part cylinder), and Mises's (all cylinder) (Table 1).

In the case of Beltrami (1885) measure, both components of energy are always used. In the case of the von Mises hypothesis, only the form energy is always used. In contrast, in the case of Huber's hypothesis, the energy form is used in the compressive stress region, and in the tensile region, the sum of volumetric and distortional internal energy $\Phi = \Phi_\nu + \Phi_f$ is used. In Fig. 7, the Beltrami, Huber, and Mises stress measures in the space of the principal axes of the stress

Table 1. Primary diagram of energy based hypothesis

Beltrami (1885)	$\Phi \leqslant K$ always	Fig. 7a
Huber (1904)	$\Phi_\nu + \Phi_f \leqslant K$ when $\sigma_1 + \sigma_1 + \sigma_3 > 0$ $\Phi_f \leqslant K$ when $\sigma_1 + \sigma_1 + \sigma_3 < 0$	Fig. 7b
Mises (1913)	$\Phi_f \leqslant K$ always	Figs. 7a,b

tensor, where the dominant is the hydrostatic axis on which the first invariant of the tensor, are depicted. For a constant critical energy K, the areas determined by the condition $f = W - K = 0$ can be obtained. In the case of the Beltrami measure, it is an ellipsoid hooked at the centre of the volume, in the case of von Mises, it is an infinite cylinder for extensions and compressions, and in the case of the Huber measure, it is the Misesian cylinder in the compression part and the Beltrami ellipsoid in the stretch part.

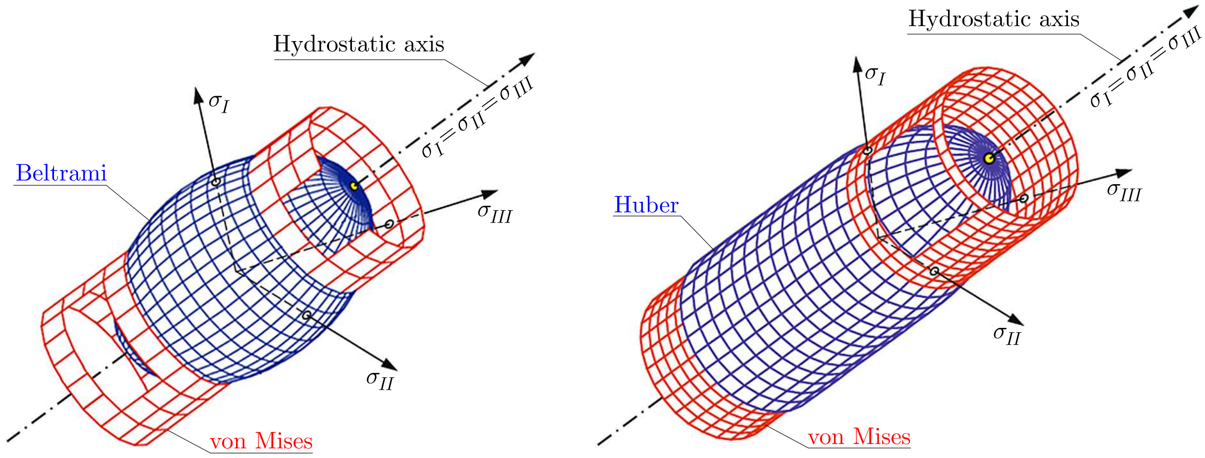

Fig. 7. Yield limits surfaces of (a) Beltrami (1884), (b) Huber (1904) and (a), (b) Mises (1913)

3.3. Revalorization of Huber's Combined Hypothesis

It was in (Schleicher, 1926) that the combined concept of Huber's effort measure was revalued for the first time after twenty-two years (Fig. 7). Schleicher, speaking about the combined Huber hypothesis, used two definitions of Huber's reduced stress, which is denoted by an additional H-index

$$\sigma_H = \begin{cases} \sqrt{2E\Phi} & \text{when} \quad I_1 < 0 \\ \sqrt{6G\Phi_f} & \text{when} \quad I_1 > 0 \end{cases} \tag{3.1}$$

Two years later, in 1928, it was criticised for the first time by Włodzimierz Burzyński (Burzyński, 1928). Huber's combined hypothesis has number C2 in his classification. The disadvantage of Huber's combined hypothesis is precisely its heterogeneity – because the states of pure stretching, two-way stretching and three-way stretching are described with full energy, and the states of single, double and three-way compression are described by form energy. Only the shear state satisfies both intervals. Burzyński, further wrote the boundary surface as shown in Fig. 8.

It shows that

$$k_s = \frac{k_t}{\sqrt{2(1+\nu)}} = \frac{k_c}{\sqrt{3}} \tag{3.2}$$

which means that the boundaries of k_t and k_c differ slightly. Using the Mohr envelope method, Burzyński showed a drawing of Huber's hypothesis (for $\nu = 1/4$) in the τ-σ plane – which

$$\left.\boxed{\begin{array}{c}\dfrac{1}{2(1+\mu)}(\sigma_1^2+\sigma_2^2+\sigma_3^2)-\dfrac{\mu}{1+\mu}(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)=k_s^2\\ \text{dla: } \sigma_1+\sigma_2+\sigma_3\geq 0 \text{ i}\\ \dfrac{1}{3}(\sigma_1^2-\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1)=k_s^2\\ \text{przy: } \sigma_1+\sigma_2+\sigma_3\leq 0\end{array}}\right\}\ \text{(C2)}$$

Fig. 8. Combined von Huber hypothesis (Burzyński, 1928), (μ is $1/\nu$, where ν is the Poisson coefficient)

nowadays is called "the Burzyński plane". It is a drawing of a rotational ellipsoid (in the Beltrami part) and a rotating cylinder (in the von Mises part) – the intersection of the two solids takes place in the planes $\sigma_1 + \sigma_2 + \sigma_3 = 0$. For $\nu = 0$, the Mises cylinder ends in a sphere, for $\nu \to 0.5$ the von Huber hypothesis becomes the von Mises hypothesis. Burzyński marks the latter as C3 and considers it separately.

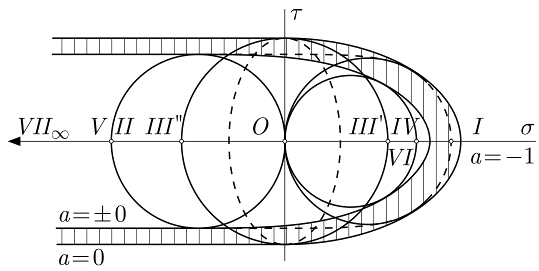

Fig. 9. Huber's cylinder-ellipsoid in the Burzyński plane (Burzyński, 1928). The Lode parameter is denoted here by letter a. By letters $I, II, III, \ldots, VII$ limit states $k_t, k_c, k_s, \ldots, k_{ccc}$ are denoted

Recently, researchers such as Altenbach, Bolchoun, Kolupaev (2016) have proposed to include Huber's hypothesis in the group of combined hypotheses. Holm Altenbach emphasizes additional advantages of Huber's hypothesis over Beltrami and Mises.

Reassuming, the most frequently used in the literature model of the J_2-plastic flow in fact possesses four different theoretical fundaments: these are (a) distortional energy source (Maxwell, 1856; Beltrami, 1885; Huber, 1904; Hencky, 1925; Schleicher, 1926; Burzyński, 1928, 1929a,b; Torre, 1947; Życzkowski, 1999), (b) von Mises norm of tensor – the second invariant of the stress deviator (Mises, 1913; Reuss, 1930; Kłębowski, 1958), (c) the stress intensity as a mean shear stress (Novozhilov, 1952), (d) the octahedral shear stress (Nadai, 1927; Zawadzki, 1954).

Only accidentally four different concepts have the same mathematical description. For instance, for the Nadai circular cone one finds

$$f = \tau_{oct}^{-2} - \frac{2}{9}(3C_0\sigma_{oct} - C_1) = 0 \tag{3.3}$$

Coming from another line of reasoning (Nadai, 1927), where τ_{oct}, σ_{oct} are octahedral shear and normal stresses and C_0, C_1 are limiting constants, it describes the same cone as the Drucker-Prager $f = I_1 - 3a_0 + 9a_1(6J_2)^{-2} = 0$ (Drucker and Prager, 1948).

It is worth noting that the Helmholtz decomposition $\varPhi = \varPhi_\nu + \varPhi_f$ also leads to decomposition of critical energy into $K = K_\nu + K_f$. Beltrami (1885) would use the word "resilience" to denote the work necessary to be done on a body to overcome its elastic forces. The volumetric (cubical) resilience K_ν is a measure of the work necessary to be expended in compression in order to increase the density permanently. Distortional resilience K_f is the work required to be expended in pure distortion in order to produce a permanent change of form in the element – it is some limit that $\varPhi_f$ can reach (Altenbach *et al.*, 2014).

3.4. Poisson coefficient dependent Schleicher's hypothesis

After many years, the concept of energy-based hypothesis of material effort was undertaken again by Schleicher (1926) who proposed to extend the Beltrami-Huber criterion $\Phi \leqslant K$ to pressure-sensitive materials: as a linear function of pressure $\sigma_{BH} = \sqrt{2E\Phi} = \sqrt{2EK}(1 - \epsilon p)$, or a parabolic function of pressure $\sigma_{BH} = \sqrt{2E\Phi} = \sqrt{2EK}(1 - \epsilon_1 p^2)$ – unfortunately, the Poisson coefficient was present in those solutions. Some response on Schleicher's paper was Burzyński's dissertation (see: Burzyński, 1928 – received to print December 13, 1927 in Lwów [now Lviv, Ukraine]), as well as in the papers (Burzyński, 1929a,b). Probably, under strong critics of Burzyński, F. Schleicher changed his measure of material effort and in the next paper, published on 13th of April, 1928 (Schleicher, 1928) he replaced σ_{BH} with the total elastic strain energy Φ by volumetric density of elastic distortion $\sigma_H = \sqrt{2E\Phi_f} = \sqrt{2EK}f(p)$. Working on brittle materials like marble, limestone, sandstone, Schleicher, finally fitted the experimental data to the cone in the $\sigma_H - p$ coordinates plane, what, in fact, was the first application of a model introduced by Drucker and Prager (1948). Note that the denotations: Beltrami-Huber σ_{BH} and Huber σ_H are originally coming from Schleicher (Dudda, 2021).

3.5. Burzyński pressure-size function hypothesis

Next, Burzyński proposed a modification for pressure-sensitive materials where the Poisson coefficient does not appear. Generally, he proposed a "size function" η_ν correcting the contribution of volumetric energy – contemporary, it is nothing else as introducing the stress triaxiality effect into the energy-based approach. It is one of the main achievements of Burzyński, since he solved a crucial question in such a way which does not disturb the scientific power of energy-based approach (Altenbach, 2010). The "size function" η_ν in the hypothesis making the volumetric energy "partially present", if $\eta_\nu = 0$ then the material is pressure-insensitive, if $\eta_\nu = 1$ then the material is fully pressure-sensitive.

Mathematically, the Burzyński hypothesis can be written as follows (Burzyński, 1928; Pęcherski, 2008; Pęcherski *et al.*, 2014)

$$\eta_\nu \Phi_\nu + \Phi_f = K \tag{3.4}$$

where a particular form of pressure dependency of the function η_ν was assumed as $\eta_\nu = \omega + \delta/p$ and ω, δ are unknown parameters. The core of Burzyński's idea is to express three unknown parameters ω, δ, K in terms of tripled of material limit constants k_c, k_t, k_s which are known from an experiment of uniaxial compression, tension and simple shear. The other forms of the size function could also be considered in order to find another state of the material like brittle and ductile failure or continuous damage. Then other experimental limits data can be used: bi-axial compression and tension k_{cc}, k_{tt}; tri-axial compression and tension k_{ccc}, k_{ttt}, and so on (Kordzikowski and Pęcherski, 2010).

3.6. The J_3 effect within the energy-based framework

Let us discuss shortly the possibility of appearance of J_3 invariant in the energy-based approach. It is a well known fact that, for common materials, the invariant J_3 (or the Lode parameter) does not appear in the elastic strain energy expression. The fundamental example is density of elastic strain energy for the so-called Hooke elastic material $\Phi = [(1-2\nu)/3E]I_1^2 + [(1+\nu)/3E]3J_2$. If such energy is a starting point to develop any independent approach, nowadays called an "energy-based" one, one can ask about a physical foundation of this. The rational arguments for defending the generality of energy-based formulations of yield conditions are the following.

Firstly, one must recall that from thermodynamical point of view (Casey and Sullivan, 1985; Enger *et al.*, 2018; Banaś and Badur, 2017) the internal energy of deformable continua depends on all intensive state parameters, not only on the elastic one. For instance, according to Taylor and Quinney (1935), the plastic deformation is also partially involved in energy Φ. According to the modern energy-based treatment, developed within the frame "strain energy density" (SED) (Łagoda and Ogonowski, 2005) the internal energy can be a function even of different internal fracture mechanisms and so on.

Secondly, papers concerning ductile failure and damage, e.g. (Bai and Wierzbicki, 2008) state that the stress triaxiality and stress shearness take part gradually, starting from initial yield surfaces completely based on classical elastic solutions. At a very low stress triaxiality state, the stress shearness (Lode parameter) has an important influence on failure behaviour. It means, that in the state of constant stress intensity, it is difficult to realize, and sophisticate damage and failure models should incorporate not-complete state of three-dimensionality of the stress state – this non-completeness, governed by the Lode parameter, is called here the stress shearness.

Speaking in terms of mathematics, the reason for adding the Lode parameter is that multiple stress state with different principal stress values can result in the same value of the stress triaxiality. Thus the stress intensity and stress triaxiality alone cannot completely describe the three-dimensional stress state and its effect on the fully developed state of damage and failure (Gao *et al.*, 2009, 2011). That means that the dimensionless invariants I_1, J_2, J_3 should be accounted in the models with plastic isotropic and kinematic hardening. On the other hand, Lou and Yoon (2017) showed that the stress shearness (the Lode parameter) had only a marginal effect on the macroscopic yield surface, whereas its influence on anisotropic damage is remarkable. Therefore, in a model that describes the starting moment of plasticity influencing the stress shearness can be neglected (Kowalczyk *et al.*, 2003).

From the above reasoning it follows that there is a physically acceptable manner of introducing J_3 into energy-based treatment. The examples can be found in Frąś *et al.* (2010, 2014), Frąś and Pęcherski (2010), Nalepka and Pęcherski (2002, 2003).

3.7. The Burzyński-Pęcherski hypothesis taking into account the third invariant

Thus, Pęcherski and his co-workers proposed to introduce the stress shearness effect into the energy-based approach and the Burzyński measure of material effort (Pęcherski *et al.*, 2011; Nowak *et al.*, 2011). Trying to find the influence of the Lode parameter, he proposed to introduce a some shape function η_f to made a partial (variable) contribution of the energy density of distortion. It means, that the extended material effort hypothesis in a case of variable energy, both with the stress triaxiality effect (volumetric energy) and the stress shearness effect (distortional energy) reads

$$\check{\eta}_\nu \Phi_\nu + \check{\eta}_f \Phi_f = K \tag{3.5}$$

where $\check{\eta}_\nu$, $\check{\eta}_f$ denote the size function and the shape function, respectively. Using the definition of Φ_ν, Φ_f in terms of invariants I_1, J_2, condition Eq. (3.5) can be expressed in the following way (Pęcherski *et al.*, 2011)

$$\eta_\nu(I_1)(3I_1)^2 + \eta_f(J_3)3J_2 = K^2 \tag{3.6}$$

The shape function (or the Lode influence function) can now be proposed in a mathematical form similar to $\mathcal{P}(\theta)$ in the maximum shearness approach. Taking, for instance, the two parameter Podgórski shape function (Podgórski, 1984)

$$\eta_f(J_3) = \frac{1}{\cos\left(\frac{\pi}{6} - \beta\right)} \cos\Big[\frac{1}{3}\arccos(\alpha\cos(3\theta)) - \beta\Big] \tag{3.7}$$

and two parametrical size function

$$\eta_\nu(I_1) = \omega + \frac{3\delta}{I_1} \tag{3.8}$$

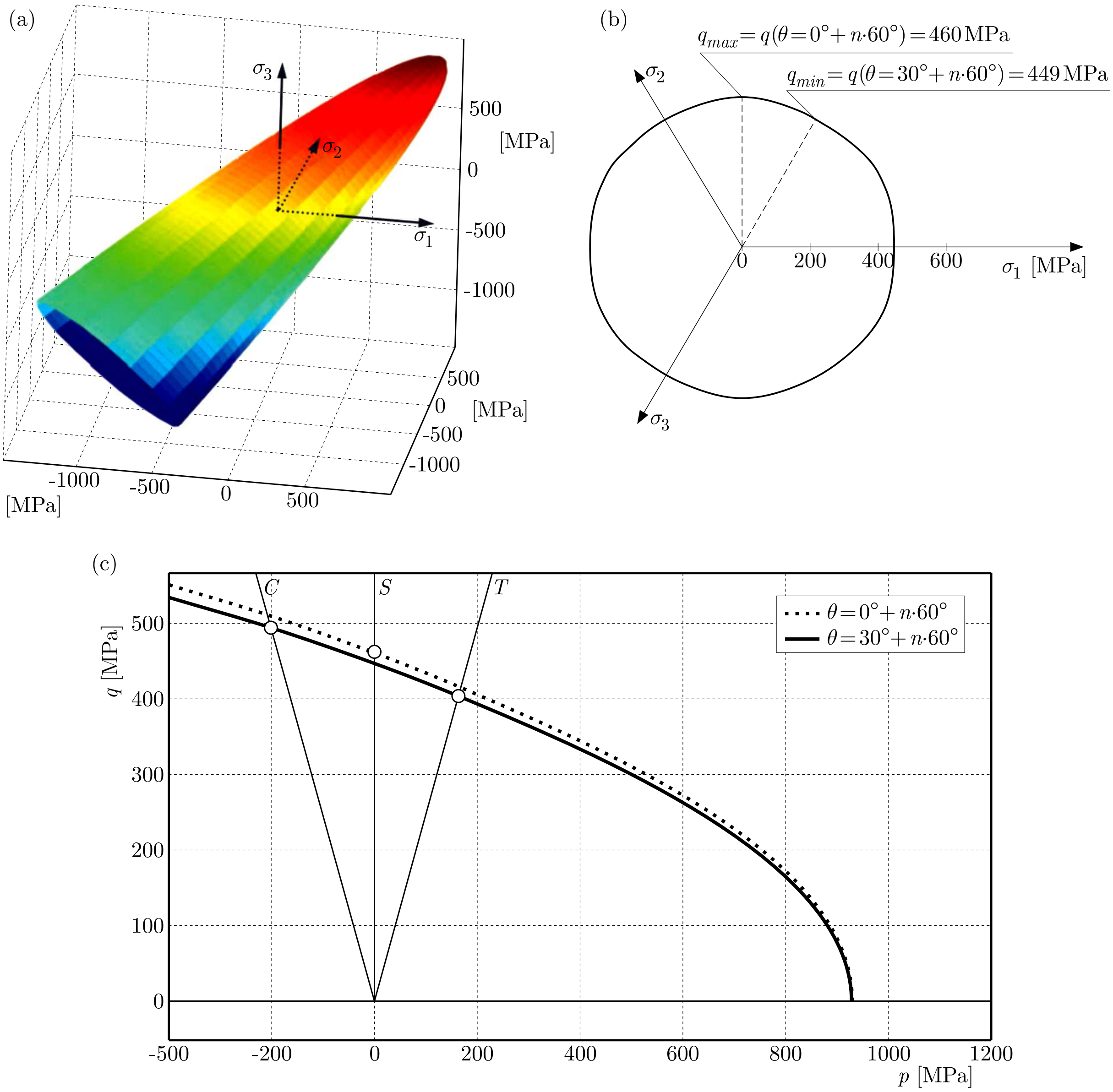

Fig. 10. The Burzyński-Pęcherski hypothesis (Pęcherski *et al.*, 2011): (a) a view to the Heigh-Westergaard cylindrical coordinates, (b) the perpendicular cross-section, (c) the Burzyński-Pęcherski hypothesis on the q-p plane

Pęcherski *et al.* (2011) obtained a five parametrical $(\alpha, \beta, \omega, \delta, K)$ yield surface, Eq. (3.5). The surface is paraboloidal and the cross-section parallel to the octahedral plane indicates that the Lode angle dependence has a hexagonal character. These five unknown parameters are to be expressed by five experimental data k_t, k_c, k_s, k_{cc}, k_{tt} – the procedure of fitting the paraboloidal yield surface, Eq. (3.6), has been made by using the Levenberg-Marquardt algorithm. Also Frąś *et al.* (2010) and Frąś and Pęcherski (2010) discussed several criteria developed within the energy-based approach in comparison with the Burzyński criterion.

The energy-like approach has some limitations (see Mucha *et al.*, 2018; Mróz and Seweryn, 1998). Let us discus a most important one coming from the fact that elastic storage energy Φ

not always is expressed in terms of J_2 invariant. For instance, the elastic energy of the gum metal is a function only of J_3 and σ_m invariants (Kowalczyk-Gajewska *et al.*, 2019).

3.8. The material anisotropy

Richard von Mises in June 1928 turned his point of view: from the Tresca-Mohr geometrical envelope approach to the energy-based approach. His aim was to construct a yield surface for generally anisotropic crystals (Mises, 1928). His mathematical concept lies on replacing $J_2 = s_{ij}s_{ji}/2$ invariant into a "weighted invariant" $J_2' = s_{ij}k_{ijkl}s_{kl}/2$ – then the anisotropic cylindrical surface is described by a dimensionless function $f = J_2' - 1 = 0$. By using many symmetry arguments, von Mises was able to reduce the number of independent coefficients of the compliance tensor k_{ijkl} to 15. It means that von Mises turn his $\tau_I^2 + \tau_{II}^2 + \tau_{III}^2 - k_s^2 = 0$ argumentation, called by him "die Guest-Mohrsche Bedingung", into energy-based argumentation.

In Section 3, von Mises, recalled the pioneering Beltrami, Huber, Schleicher, Burzyński papers, in the context of separation of the strain energy density into volumetric and distortional parts for anisotropic bodies. He proposed to start from $2\Phi = \boldsymbol{\sigma\varepsilon} = (s\mathbf{I}+\mathbf{s})\cdot(e\mathbf{I}+\mathbf{e})$ decomposition of both tensors into: axiators and deviators (Mises, 1928, p. 170).

Next, Reuss (1930) improved the von Mises idea finding a more effective method of transforming the Cauchy stress tensor from Cartesian $\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, $i, j = x, y, z$ into the principal coordinates $\boldsymbol{\sigma} = \sigma_\chi \mathbf{e}_\chi \otimes \mathbf{e}_\chi$, $\chi = 1, 2, 3$, where he reduced the number of independent coefficients to be 3 plus 3 – Euler's angles that describe the rotation matrix between the principal axes of strain and stress. Few years later, Theodore Lehmann, proposed the yield surface of a generally anisotropic material in the form of a polynomial (Lehmann, 1964)

$$f = k + k_{ij}s_{ij} + k_{ijkl}s_{ij}s_{kl} + k_{ijklmn}s_{ij}s_{kls_{mn}} + \ldots \tag{3.9}$$

Here appears some analogy to the third anisotropic invariant J_3'. If it is possible to take into account the anisotropy induced by kinematic hardening of an initially isotropic material, the above polynomial can take a form (Baltov and Sawczuk, 1965)

$$f = \frac{1}{2}s_{ij}s_{ij} - s_{ij}\alpha_{ij} + A\alpha_{ij}\alpha_{kl}(s_{ij} - \alpha_{ij})(s_{kl} - \alpha_{kl}) + \ldots \tag{3.10}$$

where α_{ij} describes traceless kinematic hardening. Additionally, let us note that the energy-based approach can be also extended into initially anisotropic materials by the concept of spectral decomposition. It is based on the Rychlewski idea introducing anisotropy between three principal stresses and three principal strains by six single stress $\boldsymbol{\sigma}_\alpha$, $\alpha = 1, 2, 3, 4, 5, 6$ (Rychlewski, 2011), determined by spectral decomposition of the compliance tensor. Ostrowska-Maciejewska *et al.* (2012, 2013) introduced a new proposition of the limit surface for anisotropic materials with asymmetry of limits, by introducing six influence functions $\eta_\alpha(\boldsymbol{\sigma}_\beta) = \eta_\alpha(I_{1(\beta)}, J_{2(\beta)}, J_{3(\beta)})$, $\alpha, \beta = 1, \ldots, 6$, into the energy density of elastic strain

$$\eta_1\Phi_1(\boldsymbol{\sigma}_1) + \ldots + \eta_\chi\Phi_\chi(\boldsymbol{\sigma}_\chi) - 1 = 0 \qquad \chi \leqslant 6 \tag{3.11}$$

This concept was applied in Szeptyński's thesis (Szeptyński, 2017) who discussed misstatements appearing in the final form of the failure criterion formulation and by Ostrowska-Maciejewska in her articles (Ostrowska-Maciejewska *et al.*, 2012, 2013) and (Kowalczyk-Gajewska and Ostrowska-Maciejewska, 2005). Recently, the energy-based approach, developed within the Burzyński framework, was extended for symmetric anisotropic materials by Moayyedian and Kadkhodayan (2017, 2021) with different application to materials exhibiting bcc, fcc and hcp crystalline structures. It prolongs the line of reasoning by Hebda and Pęcherski (2005) and Vadilo *et al.* (2011).

3.9. The week material anisotropy – four parametrical hypothesis

Some special, one-parametrical anisotropy, called by Burzyński λ-anisotropy, was developed by Pęcherski *et al.* (2021). The original (Burzyński, 1928, p. 128) solution has the following form

$$\frac{1+\nu'}{3}{\sigma'_{HMH}}^2+\frac{1}{3}(1-2\nu')(I'_1)^2+(k_c-k_t)I'_1-k_ck_t=0 \tag{3.12}$$

where "prime", similar like von Mises, means week-anisotropy modification

$$\begin{aligned} &3J'_2={\sigma'_{HMH}}^2=2(1-\lambda)(\sigma_2-\sigma_3)^2+2\lambda(\sigma_3-\sigma_1)^2+2(1-\lambda)(\sigma_1-\sigma_2)^2\\ &I'_1=3p'=\frac{\lambda\sigma_1+(1-\lambda)\sigma_2+\lambda\sigma_3}{1+\lambda}\end{aligned} \tag{3.13}$$

Here appears the "week anisotropic plasticity coefficient" $\nu'=(1/\varphi^2)(k_ck_t)/(2k_s^2)-1$, which governs the shape of limit surface (conical, cylindrical, paraboloidal, hyperbolical) also depends on λ as $\varphi^2=(2/3)(1+\lambda)$. For identification of λ, Pęcherski *et al.* (2021) proposed to make a biaxial tension test.

3.10. The week material anisotropy – many-parameter hypothesis

Similarly, starting from the energy-based arguments, Cazacu and Barlat (2004), keeping the Mises-Reuss line of reasoning, proposed a simple paraboloidal yield surface

$$3J'_2-\frac{m}{3}(I'_1)^2-1=0 \tag{3.14}$$

where invariants J'_2, I'_1 are typically calculated but for the transformed Cauchy stresses $\sigma'_{ij}=B_{ijkl}\sigma_{kl}$, where B_{ijkl} are anisotropy constants and m is a constant responsible for pressure sensitiveness. It is Burzyński's proposal, Eq. (3.10), which cannot be applied to the case with symmetry in compression and tension. For a material with transverse isotropy, like limestone, the B_{ijkl} reduces to only 5 independent components, for sn isotropic material Eq. $(3.13)_1$ reduces to the two-parameter yield surface $3J_2+(k_c-k_t)I_1-k_ck_t=0$. This model needs six data to be calibrated: compressive and tensile strengths along the transverse direction $k_{\|t}, k_{\|c}, k_{\perp t}, k_{\perp c}$ and $k_{s\|}$, $k_{s\perp}$ – the shear strength in the symmetry plane and the shear strength in the normal transversal plane. In a general case, the Cazacu-Cristescu surface, Eq. $(3.13)_1$ needs a more sophisticated set of experimental data. But, in the paper by Kowalewski (1998), the practice manner to calibrate $12+6=18$ parameters of yield condition $(3.13)_1$ for 18G2A low-alloy steel was shown. This line of reasoning can be extended for cellular materials (Kordzikowski *et al.* 2005).

4. Further comments

Here we have presented the main substantive dispute that has been and still is going on between Mohr's geometrical approach and Huber's energy approach. In this dispute, the key is the Lode experiment, which showed that for most metals the Huber method is more justified and consistent with the experimental data.

Nevertheless, as the literature shows, the geometrical approach is still preferred by the majority of researchers. The achievements of Mohr's geometrical method are documented. This method can be implemented into numerical calculations with some difficulties. As a result, it is still valid and used.

It is also easy to see that the energetic approach was developed by Huber's disciples, laying the foundation for a scientific school called "the science of effort." That is why so many Polish researchers can be found in the literature on the subject. In particular, we recall the achievements of Professor Ryszard Pęcherski (IPPT PAN, Warsaw).

But it is Holm Altenbach (Altenbach, 2010) who, preferring the energy-based approach, turns our attention on some, very emotional, scientific discussion in the triangle between William Thomson, W. Rankine, and young J. Clerk Maxwell (see: Maxwell, 1856). It was really Maxwell, who first underlined the role of distortional energy limit in fracture and plasticity. Therefore, taking Maxwell as the original author of the concept of limit distortional energy, Altenbach proposed to call it: "the Maxwell-Huber-Mises" hypothesis and to denote the equivalent stress by σ_{MHM}. In the present paper, according to the great academic tradition and with engineering experience we continue the use of "Huber-Mises-Hencky" hypothesis and σ_{HMH} denotation. This story was discussed in 1924 at the I-st International Congress of Applied Mechanics in Delft, and found a reflection in, perhaps the first, methodical elucidation of the mathematical theory of plasticity given in 1927 by H. Mierzejewski (1928). Yet other historical remarks are found in (Becchi, 1994; Altenbach, 2010; Rychlewski, 2011; Bruhns, 2014).

A historical review of the energy approach should begin with the work of Beltrami (1885). It initiated the energetic approach to the hypotheses of exertion – in particular, its development is marked by the following works:

- Beltrami (1885): $\Phi = \sigma_{ij}\varepsilon_{ji} = \Phi(\sigma_{ij}) = \Phi(\varepsilon_{ij}) \leqslant K$
- Huber (1904): $\Phi = \Phi_\nu + \Phi_f \leqslant K$ or $\Phi_f \leqslant K$
- Mises (1914), Hencky (1924): $\Phi_f \leqslant K$
- Schleicher (1926): $\Phi = \Phi_\nu(\nu) + \Phi_f(\nu) \leqslant Kf(p)$
- Burzyński (1928): $\Phi = \eta_\nu\Phi_\nu + \Phi_f \leqslant K$, $\eta_\nu = \omega + \delta/p$
- Zawadzki (1956): $\Phi = \Phi_\nu + \Phi_f + \Phi_{th} \leqslant K$
- Pęcherski (2011): $\Phi = \eta_\nu\Phi_\nu + \eta_f\Phi_f \leqslant K$, $\eta_f = 1 + \alpha[1 - e^{-\beta(1+\cos(3\theta)}]$

In the literature, the critical energy K is determined by the uniaxial tensile limit value $K = \sqrt{2Ek_t}$, $0.01 < K < 0.30\,\mathrm{MJ/m^3}$=MPa. For example, for $k_t = 700\,\mathrm{MPa}$, $E = 2.1\,\mathrm{GPa}$, we have $K = 0.23\,\mathrm{MJ/m^3}$. This line of thought, which return us to the original Huber concept expressed in Eq. (1.1), was recently re-evaluated by Nalepka and Pęcherski (2003). They proposed to calculate the critical energy K from the first principles, even those based on quantum thermodynamics. Nevertheless, in our opinion, the problem of experimental specification of the value of K is still open, and must be developed in details. Some methods developed by Ładoga and Ogonowski (2005) seem to be helpful.

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We dedicate this work to the memory of the late Professor Ryszard Pęcherski (1949-2022). Professor Pęcherski devoted many years and a lot of creative effort to renewing and revitalizing the energetic approach to the science of the material effort. In particular, Ryszard Pęcherski spent lots of time on the proper and complete understanding of Tytus Huber's works. Thanks to his job, Huber's work has been made available to the international community. Next, prof. Pęcherski took up the development of the works of Huber's student – Włodzimierz Burzyński. For many years, Professor Pęcherski, as a member of the Presidium of the Polish Society of Theoretical and Applied Mechanics, promoted the achievements of Polish mechanics.

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