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SHORT RESEARCH COMMUNICATION

RANDOM EIGENVIBRATIONS OF BEAMS WITH VISCOELASTIC LAYERS[1]

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This paper is devoted to the study of the influence of random variation of model parameters of a beam with viscoelastic layers on probabilistic characteristics of its natural frequencies and dimensionless damping coefficients. The relationships between the model parameters and the dynamic characteristics of the beam were approximated by quartic polynomials based on the results of calculations using FEM, where beam finite elements were used, taking into account lamination of the beam. The nonlinear eigenproblem was solved using the continuation method. The calculation results for an examplary laminated beam are presented.

Keywords: viscoelasticity, probabilistic characteristics, dynamic characteristics, layered beam, fractional Zener model

1. Introduction

Correct determination of basic dynamic characteristics of structures is one of the important engineering problems. However, uncertainties or variability of model parameters may have a significant impact on obtained results. Therefore, research on the influence of random variability of model parameters on probabilistic characteristics of natural vibration frequencies and dimensionless damping coefficients seems justified. This paper presents results of research conducted for a beam with layers of a viscoelastic material described by the fractional Zener model. The basic numerical tool used in this type of analysis is the Stochastic Finite Element Method – SFEM (Arregui-Mena *et al.*, 2016; Kamiński, 2013; Stefanou, 2009). To determine the probabilistic characteristics, it was necessary to perform a number of deterministic calculations for variable parameters of the beam model. The Finite Element Method was applied to these calculations, using the layered beam finite element proposed by Lewandowski and Baum (2015). This type of finite element was also used in the studies by Łasecka-Plura (2023). The dynamic characteristics of the beam were obtained by solving a nonlinear generalized eigenproblem, for which the continuation method described by Pawlak and Lewandowski (2013) was used. Research on the influence of variability of model parameters on the dynamic characteristics of plates resting on viscoelastic supports was carried out by Kamiński *et al.* (2023).

2. Finite element formulation

The dynamic analysis of the beam with viscoelastic layers was performed based on the Finite Element Method (FEM), using beam finite elements, but taking into account the layered structure

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of the beam. The finite element of the layered beam was formulated in the frequency domain. The following assumptions were made in this formulation: each viscoelastic layer is located between two elastic layers, all layers are perfectly glued to each other, the material of each layer is isotropic and homogeneous, the Euler-Bernoulli beam theory was used for elastic layers and the Timoshenko beam theory for viscoelastic layers. The corresponding layers of adjacent finite elements must have the same thickness, and damping in elastic layers is neglected. The assumptions made in this way lead to the formulation of kinematic relations for the finite element of a laminated beam, which are described in detail in the paper by Lewandowski and Baum (2015).

The fractional Zener model was chosen to describe the viscoelastic material. This model describes the actual behavior of a viscoelastic material very well, requiring only four material parameters. The constitutive equation of this model is as follows

$$\sigma(x,t)+\tau^{\alpha}D_t^{\alpha}\sigma(x,t)=E_0\varepsilon(x,t)+E_{\infty}\tau^{\alpha}D_t^{\alpha}\varepsilon(x,t) \tag{2.1}$$

This equation employs commonly used notations, namely $\sigma(x,t)$ and $\varepsilon(x,t)$ are functions of stress and strain, respectively, and τ is the relaxation time. The symbols E_0 and E_{∞} indicate the relaxed and non-relaxed elastic modules, respectively. The fractional derivative of order α is marked with the symbol $D_t^{\alpha}(\cdot)$. In the presented research, the definition of the Riemann-Louville fractional derivative was used.

After some mathematical operations, the following matrix equation of beam motion in the frequency domain can be written (Lewandowski and Baum, 2015)

$$[s^2\mathbf{M}+\mathbf{K}+\mathbf{K}_v(s)]\overline{\mathbf{q}}(s)=\overline{\mathbf{p}}(s) \tag{2.2}$$

where s is the Laplace variable, wherein $\overline{\mathbf{q}}(s)$ and $\overline{\mathbf{p}}(s)$ are the Laplace transforms of the nodal displacement vector and the external force vector, respectively. The symbols $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{K}_v(s)$ denote, respectively, the mass matrix, the stiffness matrix and the so-called viscoelastic matrix, the elements of which are functions of the variable s.

3. Dynamic characteristics of the beam with viscoelastic layers

The natural frequencies and dimensionless damping coefficients of the beam with viscoelastic layers are calculated based on the solution of the nonlinear eigenproblem, which is obtained by assuming a zero vector of the external forces in Eq. (2.2)

$$[s^2\mathbf{M}+\mathbf{K}+\mathbf{K}_v(s)]\overline{\mathbf{q}}(s)=\mathbf{0} \tag{3.1}$$

This eigenproblem can be solved by the continuation method (Pawlak and Lewandowski, 2013). Since the obtained eigenvalues are complex numbers, they can be written as

$$s_i=\mu_i+\mathrm{i}\eta_i \tag{3.2}$$

The natural frequencies and dimensionless damping coefficients can be calculated accordingly

$$\omega_i^2=\mu_i^2+\eta_i^2 \qquad\qquad \gamma_i=-\frac{\mu_i}{\omega_i} \tag{3.3}$$

4. Influence of model parameters variation on dynamic characteristics of a layered beam

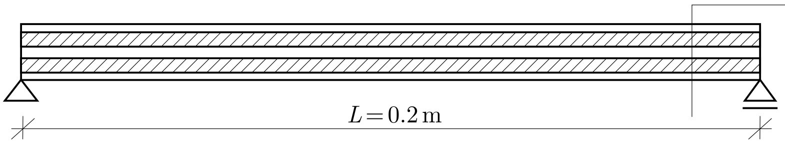
The main goal of the analyzes was to examine the impact of variability of specific parameters of the beam model on its dynamic characteristics. The analyzed characteristics are natural frequencies of the beam and dimensionless damping coefficients. In order to determine the relationship between a specific model parameter and the appropriate dynamic characteristic of the beam, a polynomial approximation using the least squares method was used based on the results of FEM calculations, i.e. a finite number of deterministic results. Quartic polynomials were adopted as approximating functions.

Inference on the influence of variability of model parameters on the natural frequencies and dimensionless damping coefficients of the laminated beam was made on the basis of the following probabilistic characteristics: expected value, standard deviation, coefficient of variation, skewness and kurtosis.

Calculations of probabilistic characteristics are, by their nature, complex tasks, which is why approximate methods are generally used in these tasks. In the conducted research, the calculation of probabilistic characteristics was carried out using three computational methods, and consistency of the obtained results was compared. The methods used were semi-analytical method – SAM (based on symbolic calculation procedures in the Maple program), stochastic perturbation technique – SPT (tenth order) and Monte-Carlo simulation – MCS (number of trials is 10^5).

5. Numerical example

This paper presents results of sample calculations performed for a beam consisting of five layers: three elastic and two viscoelastic. The model of such a beam is shown in Fig. 1.



Aluminium	$h_{e1} = 0.001\,\mathrm{m}$
Viscoelastic material	$h_{v1} = 0.002\,\mathrm{m}$
Aluminium	$h_{e2} = 0.004\,\mathrm{m}$
Viscoelastic material	$h_{v2} = 0.002\,\mathrm{m}$
Aluminium	$h_{e3} = 0.001\,\mathrm{m}$

Fig. 1. Model of an examplary layered beam

The total length of the beam is 20 cm. The lower and upper elastic layers have the same thickness, which is $h_{e1} = h_{e3} = 0.001\,\mathrm{m}$, while the thickness of the middle elastic layer is $h_{e3} = 0.004\,\mathrm{m}$. The remaining parameters of the elastic layers are as follows: Young's modulus $E_e = 70.3\,\mathrm{GPa}$, Poisson's ratio $\nu_e = 0.3$, density $\rho_e = 2690\,\mathrm{kg/m^3}$. The parameters of both viscoelastic layers are the same and their values are as follows: layer thickness $h_{v1} = h_{v2} = 0.002\,\mathrm{m}$, density $\rho_e = 1600\,\mathrm{kg/m^3}$, relaxed elastic modulus $E_0 = 1.5\,\mathrm{MPa}$, non-relaxed elastic modulus $E_\infty = 70\,\mathrm{MPa}$, Poisson's ratio – $\nu_v = 0.5$, relaxation time $\tau = 1.4 \cdot 10^{-5}\,\mathrm{s}$, parameter describing the order of derivative fractional $\alpha = 0.8$. The viscoelastic material parameters were taken from the literature (Galucio *et al.*, 2004) and describe the polymer 3M ISD112.

The analysis of the influence of variability of the parameters of the layered beam model on the probabilistic characteristics of its natural vibration frequency and dimensionless damping coefficients was performed for four parameters: thickness of the elastic layer h_{e3}, thickness of the viscoelastic layer h_{v2}, relaxation time τ and the parameter describing the order of the fractional derivative α. These parameters were treated as random variables with a Gaussian probability distribution, for which the coefficient of variation was assumed to have values ranging from 0.025 to 0.25.

As previously mentioned, the calculations were performed using three different methods: semi-analytical, perturbation technique and Monte-Carlo simulation. Figure 2 shows the results obtained using these methods. As can be seen, the graphs of the expected value, coefficient of variation, skewness and kurtosis of the first natural vibration frequency depending on the coefficient of variation of the parameter α obtained by these three methods show high agreement.

As a part of the research, a summary of the probabilistic characteristics obtained for the model parameters was made for the coefficient of variation varying within the assumed range

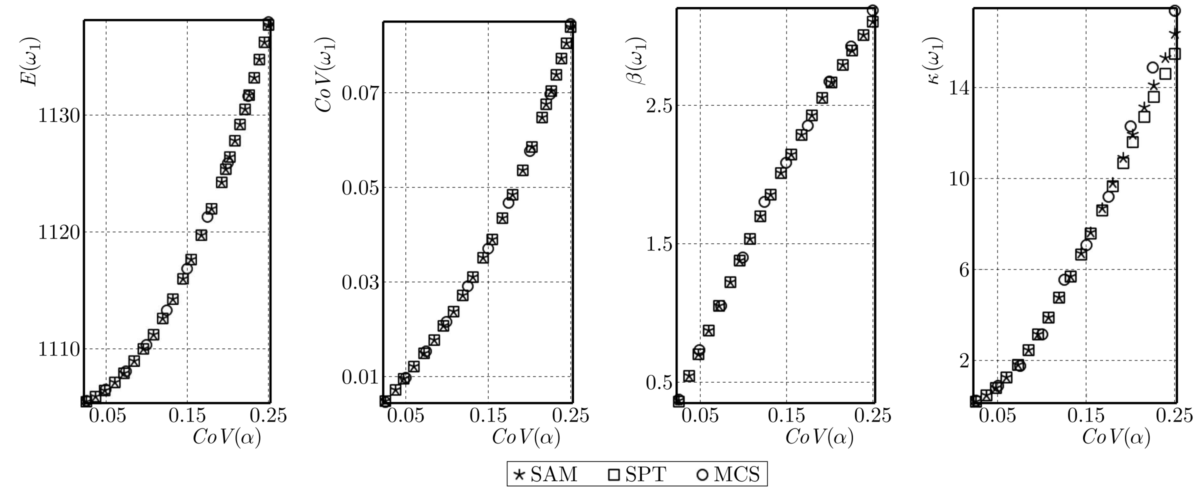


Fig. 2. Validation of methods for calculating probabilistic characteristics: expected value $E(\omega_1)$, coefficient of variation $CoV(\omega_1)$, skewness $\beta(\omega_1)$, kurtosis $\kappa(\omega_1)$ (SAM – semi-analytical method, SPT – stochastic perturbation technique, MCS – Monte Carlo simulation)

and is shown in Fig. 3. The main observation is that in the case of the first natural frequency, an increase in the variability of the parameter α has by far the greatest impact on the probabilistic characteristics. However, it should be emphasized that even in this case, the coefficient of variation of the first natural frequency ω_1 is significantly smaller than the corresponding coefficient of variation of this parameter. The impact of the variability of the remaining parameters on the probabilistic characteristics of ω_1 is significantly smaller.

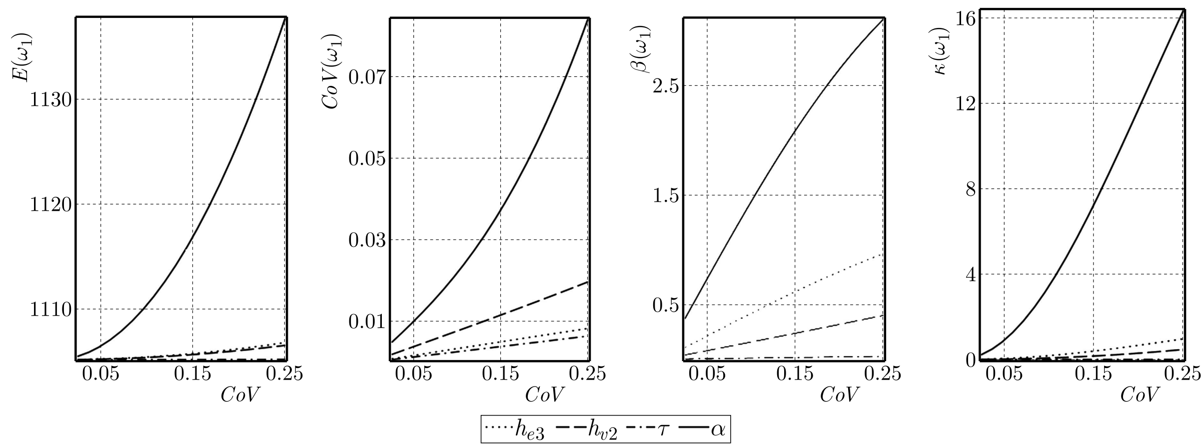


Fig. 3. Comparison of probabilistic characteristics of ω_1: expected value $E(\omega_1)$, coefficient of variation $CoV(\omega_1)$, skewness $\beta(\omega_1)$, kurtosis $\kappa(\omega_1)$ for variable coefficient of variation (CoV) of beam model parameters (results obtained by a semi-analytical method)

6. Conclusions

The calculation results presented in the previous Section allow the following conclusions to be formulated:

- Analogous analyzes as in the case of natural frequencies were performed for dimensionless damping coefficients. The results of these studies will be presented in the extended version of the article.

- Comparison of the results obtained from the semi-analytical method, stochastic perturbation technique and Monte-Carlo simulation shows that all these methods generally give similar results, but in some cases they may differ, so to be sure of the correctness of the solution, it is worth using at least two of them in parallel.
- An increase in the variability of the viscoelastic material model parameter α has by far the greatest impact on the probabilistic characteristics of natural vibration.
- Only selected research results are presented, the scope of which was much wider. In the remaining analyses, the conclusions presented above were confirmed.

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