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ASYMMETRIC DUFFING OSCILLATOR: THE BIRTH AND BUILD-UP OF A PERIOD-DOUBLING CASCADE

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We investigate the period-doubling phenomenon in aperiodically forced asymmetric Duffing oscillator. We use the known steady-state asymptotic solution – the amplitude-frequency implicit function – and known criterion for the existence of period-doubling, also in an implicit form. Working in the framework of differential properties of implicit functions, we derive analytical formulas for the birth of period-doubled solutions.

Keywords: Duffing equation, period-doubling, implicit functions

1. Introduction and motivation

A period-doubling cascade of bifurcations is a typical route to chaos in nonlinear dynamical systems. We shall study this phenomenon in Duffing-type oscillator equations.

In this work, we study period-doubling in a forced asymmetric Duffing oscillator governed by the non-dimensional equation

$$\ddot{y} + 2\zeta\dot{y} + \gamma y^3 = F_0 + F\cos(\Omega t) \tag{1.1}$$

which has a single equilibrium position and a corresponding one-well potential (Kovacic and Brennan, 2011), where ζ, γ, F_0, F are parameters and Ω is the angular frequency of the periodic force.

The period-doubling scenario in dynamical system (1.1) was investigated by Szemplińska--Stupnicka in a series of groundbreaking papers (Szemplińska-Stupnicka and Bajkowski, 1986; Szemplińska-Stupnicka, 1987, 1988), see also (Kovacic and Brennan, 2011) for a review and further results, and (Xu and Luo, 2020) for another approach to period-doubling in a similar system.

The main idea introduced in (Szemplińska-Stupnicka and Bajkowski, 1986) consists of perturbing the main steady-state (approximate) asymptotic solution of Eq. (1.1), a $1:1$ resonance $y_0(t)$

$$\begin{aligned} y(t) &= y_0(t) + B\cos\Big(\frac{1}{2}\Omega t + \varphi\Big) \\ y_0(t) &= A_0 + A_1\cos(\Omega t + \theta) \end{aligned} \tag{1.2}$$

The perturbed solution $y(t)$ is substituted into Eq. (1.1), and the condition $B \neq 0$ is demanded. In papers (Kovacic and Brennan, 2011; Szemplińska-Stupnicka and Bajkowski, 1986; Szemplińska-Stupnicka, 1987, 1988) several conditions guaranteeing the formation and stability of solution (1.2) were found and used to study the period-doubling phenomenon.

For example, these authors were able to find intervals (Ω_1, Ω_2) in which solution $(1.2)_2$ destabilized via the formation of period-doubled solution $(1.2)_1$ (Kovacic and Brennan, 2011. They also demonstrated that a cascade of period doubling leading to chaos was formed (Kovacic and Brennan, 2011; Szemplińska-Stupnicka and Bajkowski, 1986; Szemplińska-Stupnicka, 1987, 1988).

Our motivation is fueled by the observation that steady-state solution $(1.2)_2$ as well as period-doubling conditions found in (Kovacic and Brennan, 2011; Szemplińska-Stupnicka and Bajkowski, 1986; Szemplińska-Stupnicka, 1987, 1988) are in the form of some implicit functions. Therefore, it should be possible, within the framework of differential properties of implicit functions (Fikhtengolt's, 1965; Kyzioł ad Okniński, 2022), to obtain new results concerning the period-doubling mechanism.

The aim of the present work is thus to apply this formalism to implicit functions derived in (Kovacic and Brennan, 2011; Szemplińska-Stupnicka and Bajkowski, 1986; Szemplińska--Stupnicka, 1987, 1988).

The paper is organized as follows. In Section 2, a steady-state solution of Eq. (1.1) of form $(1.2)_2$ is reviewed and a period-doubling condition derived in (Kovacic and Brennan, 2011; Szemplińska-Stupnicka and Bajkowski, 1986; Szemplińska-Stupnicka, 1987, 1988) is described in Section 3. In Section 4, we derive new results concerning period-doubling applying the formalism of differential properties of implicit functions, and in Section 5 we verify our results. In Section 6, we summarize our findings.

2. The main resonance: steady-state solution

The steady-state solution of Eq. (1.1) of form $(1.3)_2$, describing the 1 : 1 resonance, was computed in (Szemplińska-Stupnicka and Bajkowski, 1986; Jordan and Smith, 1999, Kovacic and Brennan, 2011). Proceeding as in (Kyzioł and Okniński, 2023), we get two implicit equations for A_0, A_1 and Ω

$$\begin{aligned}&A_1^2\Big(3\gamma A_0^2+\frac{3}{4}\gamma A_1^2-\Omega^2\Big)^2+4\Omega^2\zeta^2A_1^2=F^2\\&\gamma A_0^3+\frac{3}{2}\gamma A_0A_1^2-F_0=0\end{aligned}\tag{2.1}$$

Computing A_1^2 from Eq. $(2.1)_2$ for $A_0 \neq 0$ and substituting into $(2.1)_1$, we obtain finally one implicit equation for A_0, Ω (Kovacic and Brennan, 2011; Kyzioł and Okniński, 2023)

$$f(A_0,\Omega,\gamma,\zeta,F_0,F)=\sum_{k=0}^{9}c_kA_0^k=0\tag{2.2}$$

where the coefficients c_k are given in Table 1 (cf. Eq. (8.3.12) in (Kovacic and Brennan, 2011).

Table 1. Coefficients c_k of polynomial (2.2)

$c_0=-F_0^3$	$c_5=4\gamma\Omega^2(\Omega^2+4\zeta^2)$
$c_1=4\Omega^2F_0^2$	$c_6=-15\gamma^2F_0$
$c_2=-4F_0\Omega^4-16\zeta^2\Omega^2F_0$	$c_7=-20\Omega^2\gamma^2$
$c_3=-9\gamma F_0^2+6\gamma F^2$	$c_8=0$
$c_4=16\Omega^2\gamma F_0$	$c_9=25\gamma^3$

We can also obtain an implicit equation for A_1, Ω as in (Kyzioł and Okniński, 2023). Solving Eq.$(2.1)_2$ for A_0 (there is only one real root) and substituting to $(2.1)_1$, we get

$$g(A_1,\Omega,\gamma,\zeta,F,F_0)=A_1^2\Big(3\gamma A_0^2+\frac{3}{4}\gamma A_1^2-\Omega^2\Big)^2+4\Omega^2\zeta^2A_1^2-F^2=0\tag{2.3}$$

where A_0 and Y are defined as

$$A_0 = -\frac{A_1^2}{2Y} + Y \qquad Y = \sqrt[3]{\sqrt{\frac{1}{8}A_1^6 + \frac{1}{4\gamma^2}F_0^2} + \frac{1}{2\gamma}F_0} \tag{2.4}$$

3. Birth of period-doubling

The stability of the steady-state solution $y_0(t) = A_0 + A_1\cos(\Omega t + \theta)$ is studied via substitution into Eq. (1.1): $y = y_0 + u(t)$, with $u(t)$ small. This substitution leads to (linear) Hill's equation for $u(t)$ (Kovacic and Brennan, 2011; Szemplińska-Stupnicka and Bajkowski, 1986)

$$\begin{aligned} &\widehat{L}u \equiv \ddot{u} + 2\zeta\dot{u} + [\sigma_0 + \sigma_1\cos(\Omega t + \theta) + \sigma_2\cos(2(\Omega t + \theta))]u = 0 \\ &\sigma_0 = 3\gamma A_0^2 + \frac{3}{2}\gamma A_1^2 \qquad \sigma_1 = 6\gamma A_0 A_1 \qquad \sigma_2 = \frac{3}{2}\gamma A_1^2 \end{aligned} \tag{3.1}$$

provided that higher powers of u are neglected.

To study destabilization of the 1 : 1 resonance via the period-doubling scenario, one puts into Eq. (3.1)

$$u(t) = B\cos\Big(\frac{1}{2}\Omega t + \varphi\Big) \tag{3.2}$$

obtaining, after a harmonic balance method is used, a simple necessary condition for the onset of period-doubling (a condition for non-zero B)

$$h(A_0, A_1, \Omega; \gamma, \zeta) \equiv \Big(\sigma_0 - \frac{1}{4}\Omega^2\Big)^2 + \zeta^2\Omega^2 - \frac{1}{4}\sigma_1^2 = 0 \tag{3.3}$$

see Eq. (8.5.5) in (Kovacic and Brennan, 2011) or Eq. (4d) in (Szemplińska-Stupnicka and Bajkowski, 1986).

Equation (3.3) can be simplified. We compute A_1^2 from Eq. $(2.1)_2$ for $A_0 \neq 0$ and substitute it into Eq. (3.3), obtaining a simplified condition for the birth of period-doubling

$$\begin{aligned} k(A_0, \Omega; \gamma, \zeta, F_0) &= 10\gamma^2 A_0^6 - \gamma\Omega^2 A_0^4 - 2\gamma F_0 A_0^3 \\ &+ \Big(\frac{1}{16}\Omega^4 + \zeta^2\Omega^2\Big)A_0^2 - \frac{1}{2}F_0\Omega^2 A_0 + F_0^2 = 0 \end{aligned} \tag{3.4}$$

4. Differential condition for period-doubling

We are going to show that there is a differential condition that permits a further insight into the nature of the birth of period doubling.

We consider a quite obvious differential condition for the birth of period-doubling. More precisely, we investigate when equations (2.2), (3.4) have a common (real) root (A_0, Ω). Equation (2.2) guarantees that an isolated point of implicit function (3.4) lies in the 1 : 1 resonance amplitude-frequency curve (2.2) – a double solution of Eqs. (2.2), (3.4). This corresponds to a birth of instability of the 1 : 1 resonance with the creation of period-doubled solution (3.2), i.e. to a birth of period-doubling.

Accordingly, we consider the following equations

$$\begin{aligned} &f(A_0, \Omega; \gamma, \zeta, F_0, F) = 0 \qquad k(A_0, \Omega; \gamma, \zeta, F_0) = 0 \\ &\frac{\partial k(A_0, \Omega, \gamma, \zeta, F_0)}{\partial A_0} = 0 \qquad \frac{\partial k(A_0, \Omega, \gamma, \zeta, F_0)}{\partial \Omega} = 0 \end{aligned} \tag{4.1}$$

where Eq. $(4.1)_1$, equivalent to (2.2), is the steady-state condition for the 1 : 1 resonance, Eq. $(4.1)_2$ is period-doubling condition (3.4) and equations $(4.1)_{3,4}$ mean that implicit function (3.4) has a singular point. We show below that this singular point is an isolated point of (3.4).

Acceptable solutions of Eqs. (4.1), i.e $\Omega > 0$, $A_0 > 0$, $\gamma > 0$, $F > 0$ are

$$\Omega_* = c_1\zeta \qquad A_{0*} = c_2\frac{F_0}{\zeta^2} \qquad \gamma_* = c_3\frac{\zeta^6}{F_0^2} \qquad F_* = c_4 F_0 \tag{4.2}$$

and

$$\begin{aligned} c_1 &= 2\sqrt{2}\sqrt{\sqrt{2}+1} \cong 4.395 \qquad & c_2 &= \frac{3}{4}\Big(1-\frac{1}{2}\sqrt{2}\Big) \cong 0.220 \\ c_3 &= \frac{32}{27}(7\sqrt{2}+10) \cong 23.585 \qquad & c_4 &= \frac{3}{8}\sqrt{2}\sqrt{16\sqrt{2}+73} \cong 5.186 \end{aligned} \tag{4.3}$$

where $\zeta > 0$, $F_0 > 0$ are free parameters. These solutions have been computed using Maple from Scientific WorkPlace 4.0. Finally, we compute A_{1*}^2 from Eqs. $(2.1)_2$, (4.2) and (4.3) and check that inequality $A_{1*}^2 > 0$ is fulfilled for $F_0 > 0$ since $(3/2)\gamma A_{0*}A_{1*}^2 = F_0 - \gamma_* A_{0*}^3 = (3/4)F_0 > 0$.

Now we demonstrate that solution (4.2) and (4.3) corresponds to an isolated point. The determinant of the Hessian matrix, computed for the function $k(A_0, \Omega, \zeta, \gamma, F_0 t)$ at singular point (4.2) and(4.3), is positive

$$\det\begin{bmatrix} \dfrac{\partial^2 k(A_0,\Omega)}{\partial\Omega^2} & \dfrac{\partial^2 k(A_0,\Omega}{\partial\Omega\partial A_0} \\ \dfrac{\partial^2 k(A_0,\Omega)}{\partial A_0\partial\Omega} & \dfrac{\partial^2 k(A_0,\Omega)}{\partial A_0^2} \end{bmatrix} = (36\sqrt{2}+18)\zeta^2 F_0^2 > 0 \tag{4.4}$$

and this means that this is an isolated point (Fikhtengolt's, 1965).

For example, if we choose $\gamma = 0.1$ and $F_0 = 0.02$ then we compute from Eqs. (4.2) and (4.3) other parameters of the isolated point, ζ_*, F_*, as well as Ω_*, A_{0*}, and A_{1*} from Eq. $(2.1)_2$, listed in Table 2.

Table 2. Variables and parameters computed from Eqs. (4.2) and (4.3), $\gamma = 0.1$, $F_0 = 0.02$

ζ_*	γ	F_0	F_*	Ω_*	A_{0*}	A_{1*}
0.109204	0.1	0.02	0.103721	0.479923	0.368403	0.521001

And indeed, if we solve Eqs. $(4.1)_{1,2}$ for this set of parameters, we obtain a double solution $(A_{0*}, \Omega_*) = (0.368403, 0.479923)$, see also Fig. 1, where the singular point (A_{0*}, Ω_*) – an isolated point of implicit function (3.4) – is shown as a red dot.

For $\zeta > \zeta_*$ solutions of Eq. $(4.1)_2$ are complex, for $\zeta = \zeta_*$ there is a real isolated point (a double root of Eqs. (4.1)) lying in curve $(4.1)_1$ – a red dot in Fig. 1, and for decreasing values of ζ curves $(4.1)_2$ are growing blue ovals. More exactly, in Fig. 1 we have $\gamma = 0.1$,$F_0 = 0.02$, $F = 0.103721$ while $\zeta = 0.109204$ (singular – isolated point), 0.109, 0.108, 0.105, 0.100.

Note that for decreasing ζ implicit function $(4.1)_1$, describing the 1 : 1 resonance changes only slightly while implicit function $(4.1)_2$, destabilization condition of the resonance, changes significantly.

5. Numerical verification

It follows from Section 4 that for $\gamma = 0.1$, $F_0 = 0.02$, $F = 0.103721$, destabilization of the 1 : 1 resonance occurs for $\zeta \leqslant \zeta_* = 0.109204$. Therefore, we have computed bifurcation diagrams solving Eq. (1.1) for $\gamma = 0.1$, $F_0 = 0.02$, $F = 0.103721$, and $\zeta \approx 0.109204$ looking for an onset of period-doubling. And indeed, the 1 : 1 resonance becomes unstable for $\zeta \cong 0.1105$.

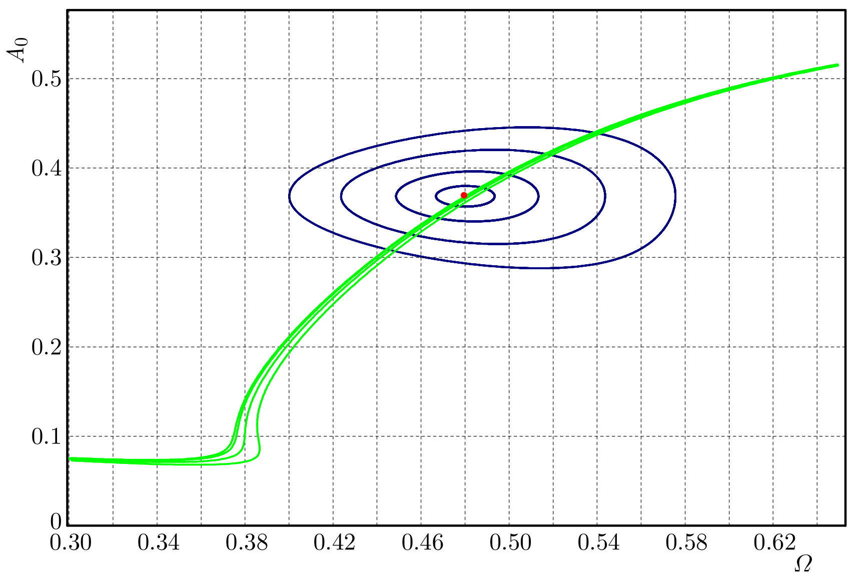

Fig. 1. Implicit functions: $(4.1)_1$ (green), $(4.1)_2$ (blue), isolated point (red); $\gamma = 0.1$, $F_0 = 0.02$, $F = 0.103721$ and $\zeta = 0.109204$ (singular), 0.109, 0.108, 0.105, 0.100

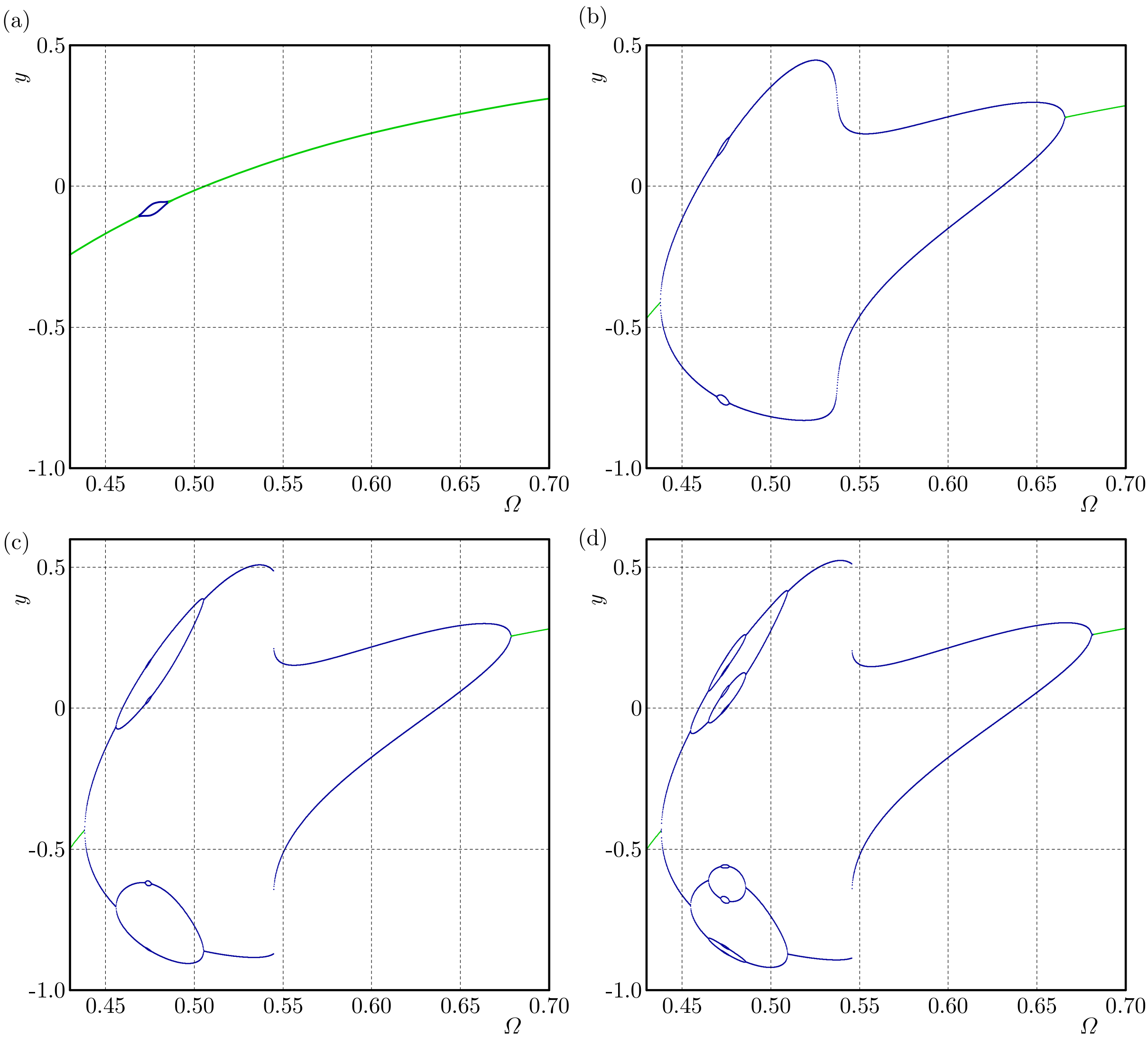

Fig. 2. Bifurcation diagrams: y against Ω, $\gamma = 0.1$, $F_0 = 0.02$, $F = 0.103721$ and $\zeta = 0.1105$, 0.0621, 0.0562, 0.0550 in diagrams 1, 2, 3, 4, respectively; 1 : 1 (green) and period-doubled (blue) resonances

Numerical solutions of $y(t)$ from Eq. (1.1) (bifurcation diagrams) were computed running DYNAMICS (Nusse and York, 1998) and Wolfram Mathematica 12.1 (Wolfram, 2020) in the interval $\Omega \in (0.43, 0.70)$ for $\gamma = 0.1$, $F_0 = 0.02$, $F = 0.103721$ and $\zeta = 0.1105$, 0.0621, 0.0562, 0.0550, see Fig. 2. The initial conditions were $y(0) = 0$, $\dot{y}(0) = 0$.

We note that destabilization of the 1 : 1 resonance with the formation of 1 : 2 solution (1.2) (as well as other resonances) appears at $\Omega = 0.47\ldots$ in a good agreement with the analytical value $\Omega_* = 0.479923$ in Table 2.

Moreover, we have computed numerical values of parameter ζ at which the first and subsequent period-doubling bifurcations occur, see Table 3. More precisely, we have been computing bifurcation diagrams for decreasing values of ζ until the period-doubled solution was visible in the corresponding bifurcation diagram (see, for example, diagram 1 in Fig. 2). The first period-doubling takes place at $\zeta_1 = 0.1105334$ in good agreement with analytical value $\zeta_* = 0.109204$, see Table 2. We have also computed ratios $(\zeta_{i-1} - \zeta_{i-2})/(\zeta_i - \zeta_{i-1})$ which converge quite well to the Feigenbaum constant $\delta = 4.6692011609\ldots$ (Feigenbaum, 1978).

Table 3. Period-doubling bifurcations, $\gamma = 0.1$, $F_0 = 0.02$, $F = 0.103721$

i	period 2^i	Ω_i	ζ_i	$\dfrac{\zeta_{i-1} - \zeta_{i-2}}{\zeta_i - \zeta_{i-1}}$
1	2^1	0.477000	0.1105334	–
2	2^2	0.473200	0.0622045	–
3	2^3	0.474200	0.0562256	8.083
4	2^4	0.474550	0.0550668	5.160
5	2^5	0.474515	0.0548240	4.772
6	2^6	0.474515	0.0547719	4.660

6. Summary

Based on the known amplitude-frequency steady-state equation for 1 : 1 resonance (2.2) and period-doubling condition (3.4), we have computed a two-parameter family of solutions of Eqs. (4.1) – see Eqs. (4.2) and(4.3). Equations $(4.1)_2, 3, 4$ guarantee that these solutions are singular points of period-doubling condition (3.4) while Eq. $(4.1)_1$, equivalent to Eq. (2.2), entails that solutions of equations (4.1) fulfill also the 1 : 1 resonance condition.

We have demonstrated that these singular points are isolated points (cf. Eq. (4.4)) which due to Eq. $(4.1)_1$ lie in the amplitude-frequency curves of the 1 : 1 resonance as in Fig. 1. The emergence of an isolated point, a double real root of Eqs. (4.1), corresponds thus to the onset of period-doubling.

Furthermore, we have obtained a good agreement between analytical value $\zeta = \zeta_* = 0.109204$, Table 2, and numerical value $\zeta = \zeta_1 = 0.1105334$ for the onset of period-doubling, Table 3. Moreover, the first period-doubling occurs for $\Omega_1 = 0.477000$ in agreement with analytical value $\Omega_* = 0.479923$.

It is possible to control destabilization of the 1 : 1 resonance by decreasing ζ, $\zeta < \zeta_*$. Indeed, it follows from Fig. 2 and Table 3 that upon decreasing ζ, we observe a build-up of the Feigenbaum cascade of period-doubling, leading to chaos. We note that all period-doubling occur for $\Omega \in (0.47, 0.48)$, and this suggests that in the case of higher period-doubling there is a similar mechanism at work.

We hope, that our approach can be applied to other non-linear dynamical systems – periodically forced nonlinear ordinary differential equations.

A. Computational details

Nonlinear polynomial equations (4.1) were solved using the computational engine Maple 4.0 from Scientific WorkPlace 4.0. Figure 1 was plotted with the computational engine MuPAD 4.0 from Scientific WorkPlace 5.5. Bifurcation diagrams in Fig. 2 were computed by integrating numerically Eq. (1.1) running DYNAMICS (Nusse and York, 1998) and Wolfram Mathematica 12.1 (Wolfram, 2020).

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