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TRANSIENT VIBRATION OF A FRACTIONAL VISCOELASTIC CANTILEVER BEAM WITH AN ECCENTRIC MASS ELEMENT AT THE END[1]

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The work focuses on the transient forced vibration of a cantilever beam with a rigid eccentric mass element attached at the free end. The Euler-Bernoulli beam theory and the viscoelastic fractional Kelvin-Voigt material model are adopted. The equation of motion of the beam is derived using Hamilton's principle. The first eigenfunction of linear vibrations is used as an approximate solution for the nonlinear vibrations. The equations of motion of the system are solved numerically. The impact of the order of the fractional derivative on the beam transient linear and nonlinear vibrations is studied.

Keywords: fractional viscoelasticity, beam vibration, transient dynamic analysis, nonlinear vibrations

1. Introduction

Cantilever beams with a tip mass element are commonly used to model various engineering structures, such as tall buildings, offshore structures, moving cranes, masts, accelerometers, military airplane wings, accelerometers, Stockbridge dampers, energy harvesters, turbine blades (Rama Bhat and Wagner, 1976; Erturk and Inman, 2011; Gürgöze and Zeren, 2011; Markiewicz, 1995; Seidel and Csepregi, 1984). This issue has been studied extensively by many researchers for different variants of cantilever beams (Gürgöze and Zeren, 2011; Suzuki *et al.*, 2021; Yang, 2017). However, many studies have omitted some important issues, such as the effects of material damping and eccentricity on system dynamics. It is rather obvious that in some vibration studies of such beam systems it is necessary to take eccentricity into account, namely that the center of mass of the element does not coincide with its point of attachment to the end of the beam. This eccentricity can affect dynamic properties of the analyzed system (Gürgöze and Zeren, 2011; Suzuki *et al.*, 2021; Yang, 2017). Similarly, viscoelastic properties of the material may significantly affect the dynamic behavior of the system, thus a proper viscoelastic material model should be used in dynamic analysis. Some experiments revealed that numerous engineering materials show a weak frequency dependence of their damping properties within a wide frequency range (Torvik and Bagley, 1984; Caputo, 1967). The description of this feature is complicated, and it is usually performed with the help of integer order derivatives (Caputo, 1967).

In recent decades, fractional calculus has been increasingly used in many scientific researches. Fractional derivatives are widely utilized in mechanics of materials, control systems, mechatronics, thermoelasticity, signal and image processing, engineering biology and many other (Freundlich, 2016; Podlubny, 1999; Shen *et al.*, 2022; Sumelka, 2016; Sumelka *et al.*, 2020; Tayel and Hassan, 2019). Since, fractional derivatives are not local, they are used for modeling of non-local phenomena i.e. depending on the history process, therefore fractional derivatives are widely used in a description of viscoelastic material behavior (Torvik and Bagley, 1984; Rossikhin and

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Shitikova, 2009). Fractional derivatives allows more accurate modeling of viscoelastic material behavior in a wide range of frequencies (Torvik and Bagley, 1984; Caputo, 1967).

This paper presents a study of transient vibrations of a fractional cantilever beam with a rigid mass element attached at its free end, whose center of mass does not coincide with the point of its attachment. The study is a continuation and extension of the author's earlier works (Freundlich, 2019, 2021). The mentioned works have been focused on vibration of a fractional viscoelastic cantilever beam with an end mass element, whose center of gravity coincides with the point of its attachment. In the first mentioned work, a fractional viscoelastic Kelvin-Voigt material model was used (Freundlich, 2019), whereas the second work adopted a fractional viscoelastic Zener material model (Freundlich, 2021). Therefore, this study is dedicated to transient dynamic analysis of a cantilever beam having the end mass element, whose center of gravity is not coincident with the point of its attachment.

2. Problem formulation

In this work, dynamic analysis of a homogeneous cantilever beam of length l having an eccentric heavy element of mass m_p and moment of inertia I_B, which is attached at the beam free end is presented. The mass center of the mass element does not coincide with the free end of the beam, and there is an eccentricity of distance e (Fig. 1). The analyzed beam has uniform cross-

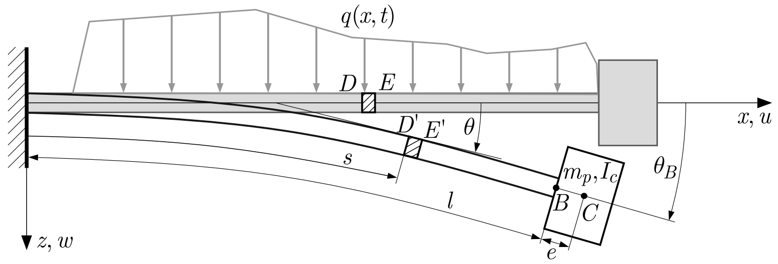

Fig. 1. Schematic of the analyzed beam

-section A and mass density ρ. The case of thin inextensible beam subjected to large deformation is studied. The Euler-Bernoulli theory is assumed, namely, that the rotary inertia and shear deformation are neglected. Moreover, the beam motion is assumed only in the xz-plane and that the gravitational force is perpendicular to this plane, thus the gravitational force has no effect on the beam motion. The viscoelastic beam material properties are assumed to be described using a fractional Kelvin-Voigt model, which is defined below (Torvik and Bagley, 1984; Mainardi and Spada, 2011)

$$\sigma(t) = E(\varepsilon(t) + \mu_\gamma D^{(\gamma)}(\varepsilon(t))) \tag{2.1}$$

where $\sigma(t)$ and $\varepsilon(t)$ are the stress and strain functions of time, μ_γ is a time constant (Mainardi and Spada, 2011), E is the relaxed modulus, t is time and $D^{(\gamma)}(\cdot)$ is the Caputo fractional derivative of the order γ, formulated as Eq. (2.2) (Caputo, 1967; Mainardi and Spada, 2011; Podlubny, 1999). For integer order derivative i.e. $\gamma = 1.0$, time constant μ_γ reduces to retardation time of the classical Kelvin-Voigt material. The unit of μ_γ is s^γ

$$D^{(\gamma)}(f(t)) \equiv \frac{d^\gamma}{dt^\gamma}(f(t)) \equiv \frac{1}{\Gamma(\mathfrak{M} - \gamma)} \int\limits_0^t \frac{D^{(\mathfrak{M})}(f(\tau))}{(t-\tau)^{\gamma+1-\mathfrak{M}}}\, d\tau \tag{2.2}$$

where $\Gamma(\mathfrak{M} - \gamma)$ is the Euler gamma function (Podlubny, 1999), $D^{(\mathfrak{M})}(f(\cdot)) = (\partial^{\mathfrak{M}}/\partial t^{\mathfrak{M}})(\cdot)$ is the $\mathfrak{M}$-th derivative of a function $f(\cdot)$ with respect to time, $\mathfrak{M}$ is a positive integer number satisfying the inequality $\mathfrak{M} - 1 < \gamma < \mathfrak{M}$, and $t > 0$.

In the case of dissipative forces, the value of γ is assumed to be in the range $0 < \gamma \leqslant 2$ (Malendowski and Sumelka, 2023), however for many real viscoelastic materials, the order of the fractional derivative is often assumed to be in the range $0 < \gamma \leqslant 1$ (Torvik and Bagley, 1984; Caputo, 1967) and then $\mathfrak{M} = 1$. Eq. (2.2), where $\gamma = 1.0$ is in the case of an integer order derivative (Mainardi and Spada, 2011; Podlubny, 1999).

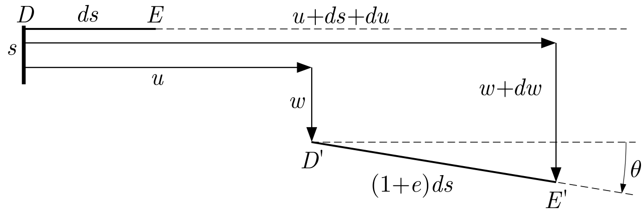

Fig. 2. Schematic of a beam displacements

The kinematics of the considered beam is presented in Fig. 2. From kinematic analysis it follows that

$$r d\theta = ds \;\longrightarrow\; \frac{1}{r} = \kappa = \frac{\partial \theta}{\partial s} = \theta' \qquad\qquad \sin\theta = \frac{\partial w}{\partial s} = w' \;\Rightarrow\; \theta = \arcsin w' \tag{2.3}$$

where r is radius of curvature, κ is curvature.

Since the beam is assumed to be inextensible, therefore

$$\cos\theta = \frac{u + ds + du - u}{(1+\mathfrak{e})ds} = \frac{ds + du}{(1+\mathfrak{e})ds} = \frac{1+u'}{(1+\mathfrak{e})} \qquad \text{for} \qquad \mathfrak{e} = 0 \qquad \cos\theta = 1 + u' \tag{2.4}$$

then

$$\kappa = \theta' = \frac{\partial}{\partial s}(\arcsin w') = \frac{1}{\sqrt{1-(w')^2}} w'' \approx w''\Big(1 + \frac{1}{2}(w')^2\Big) \tag{2.5}$$

From the beam theory it follows that the strain is

$$\varepsilon(t) = -z\kappa = -zw''\Big(1 + \frac{1}{2}(w')^2\Big) \tag{2.6}$$

The extended Hamilton principle is utilized to obtain the equation of motion (Meirovitch, 1967)

$$\int\limits_{t_1}^{t_2} (\delta T - \delta \mathit{\Pi} + \delta L) = 0 \tag{2.7}$$

where: δT and $\delta \mathit{\Pi}$ are variations of the system total kinetic and potential energy, respectively, δL is the total virtual work done by non-conservative forces.

The total kinetic energy of the system is the sum of kinetic energy of the beam and the end mass element. From literature it follows that the impact of beam longitudinal velocity u on the total system kinetic energy can be omitted, thus the total system kinetic energy is expressed as

$$T = \frac{1}{2}\int\limits_0^l m\dot{w}^2\, dx + \frac{1}{2} m_p \dot{w}_C^2 + \frac{1}{2} I_C (\dot{\theta}_C)^2 \tag{2.8}$$

where over-dots $(\dot{\cdot})$ and $(\ddot{\cdot})$ mean the first and second derivatives with respect to time, m is mass density per unit length, $\dot{w}$ is velocity of the neutral beam axis point in z direction, $\dot{w}_C$ is velocity of the center of mass C of the end mass element, $\dot{\theta}_C$ is angular velocity of the end mass element

(Fig. 1). Substituting the expression for time derivative of angle θ Eq. (2.3) into Eq. (2.8), the kinetic energy reads

$$\begin{aligned} T = \frac{1}{2}\int\limits_0^l m\dot{w}^2\,dx + \frac{1}{2}m_p\Big[\Big(-\frac{1}{\sqrt{1-(w_B')^2}}\dot{w}_B' e\sin\theta_B\Big)^2 \\ + \Big(\dot{w}_B + \frac{1}{\sqrt{1-(w_B')^2}}\dot{w}_B' e\cos\theta_B\Big)^2\Big] + \frac{1}{2}I_C\Big(\frac{1}{\sqrt{1-(w_B')^2}}\dot{w}_B'\Big)^2 \end{aligned} \tag{2.9}$$

Using an approximate relationship

$$\frac{1}{\sqrt{1-(w')^2}} \approx 1 + \frac{1}{2}(w')^2$$

the variation of the system kinetic energy is expressed as

$$\delta T = \delta\Big(\frac{1}{2}\int\limits_0^l m\dot{w}^2\,ds + \frac{1}{2}(m_p e^2 + I_C)(\dot{w}_B')^2(1+(w_B')^2) + \frac{1}{2}m_p(\dot{w}_B^2 + 2\dot{w}_B\dot{w}_B' e)\Big) \tag{2.10}$$

From the assumed beam model it follows that the total potential energy is the strain energy of the beam. Utilizing relations (2.5) and (2.6), the variation of the beam strain energy can be expressed as

$$\begin{aligned} \Pi_b = \frac{1}{2}\int\limits_A\int\limits_0^l E\varepsilon^2\,dA\,dx = \frac{1}{2}\int\limits_A\int\limits_0^l E(z)^2\kappa^2\,dA\,dx = \frac{1}{2}\int\limits_0^l EJ\kappa^2\,dx \\ \Rightarrow\quad \delta\Pi_b = \delta\Big(\frac{1}{2}\int\limits_0^l EJ\frac{(w'')^2}{1-(w')^2}\,dx\Big) \approx \delta\Big(\frac{1}{2}\int\limits_0^l EJ[(w')^2 + (w'')^2(w')^2]\,dx\Big) \end{aligned} \tag{2.11}$$

where E is Young's modulus of the beam material, A is cross-section area of the beam, J is cross-section moment of inertia with respect to the neutral beam axis.

Virtual work of non-conservative forces is a sum of work done by internal dissipation forces and external forces acting on the beam, namely

$$\begin{aligned} \delta L_{nc} &= -\int\limits_0^l\int\limits_A \sigma_{dis}\delta\varepsilon\,dA + \int\limits_0^l q\delta w\,ds \\ &= -E_\gamma' J\int\limits_0^l \frac{d^\gamma}{dt^\gamma}\Big[w''\Big(1+\frac{1}{2}(w')^2\Big)\Big]\Big(1+\frac{1}{2}(w')^2\Big)\delta(w'')\,ds \\ &\quad - E_\gamma' J\int\limits_0^l \frac{d^\gamma}{dt^\gamma}\Big[w''\Big(1+\frac{1}{2}(w')^2\Big)\Big]w''w'\delta(w')\,ds + \int\limits_0^l q\delta w\,ds \end{aligned} \tag{2.12}$$

Substituting the expanded and transformed expression for variations of kinetic energy (Eq. (2.10)), strain potential energy (Eq. (2.11)), and virtual work (Eq. (2.12)) into Hamilton's extended principle Eq. (2.7), the following equation of motion and boundary conditions of the analyzed system is obtained

$$\begin{aligned} & m\ddot{w} + EJw'''' + EJ[w''''(w')^2 + 6w''w'''w' + 3(w'')^3] \\ & \quad + E_\gamma' J\Big\{\frac{d^\gamma}{dt^\gamma}\Big[w''''\Big(1+\frac{1}{2}(w')^2\Big) + 3w''w'''w' + (w'')^3\Big]\Big(1+\frac{1}{2}(w')^2\Big)\Big\} \\ & \quad + E_\gamma' J\frac{d^\gamma}{dt^\gamma}\Big[w'''\Big(1+\frac{1}{2}(w')^2\Big) + (w'')^2 w'\Big]w'w'' = q \end{aligned} \tag{2.13}$$

Boundary conditions are obtained directly from Hamilton's principle, and for $s = 0$, the beam deflection and slope equals 0, thus

$$w = w' = 0 \tag{2.14}$$

Whereas, the boundary conditions for $s = l$ are as follows

$$\begin{aligned}
&- m_p\ddot{w}_B - m_p\ddot{w}'_B e + EJ[w''' + w'''(w')^2 + 2w'(w'')^2 - (w'')^2w'] \\
&\quad + E'_\gamma J\Big\{\frac{d^\gamma}{dt^\gamma}\Big[w'''\Big(1+\frac{1}{2}(w')^2\Big) + (w'')^2w'\Big]\Big(1+\frac{1}{2}(w')^2\Big)\Big\} \\
&\quad + E'_\gamma J\Big\{\frac{d^\gamma}{dt^\gamma}\Big[w''\Big(1+\frac{1}{2}(w')^2\Big)\Big]w'w'' - \frac{d^\gamma}{dt^\gamma}\Big[w''\Big(1+\frac{1}{2}(w')^2\Big)\Big]w''w'\Big\} = 0 \\
&- (m_pe^2 + J_C)[\ddot{w}'_B(1+(w'_B)^2) + 2(\dot{w}'_B)^2w'_B] - m_p\ddot{w}_Be - EJ(w'' + w''(w')^2) \\
&\quad - E'_\gamma J\Big\{\frac{d^\gamma}{dt^\gamma}\Big[w''\Big(1+\frac{1}{2}(w')^2\Big)\Big]\Big(1+\frac{1}{2}(w')^2\Big)\Big\} = 0
\end{aligned} \tag{2.15}$$

By introducing dimensionless parameters

$$\begin{aligned}
&\tau = \sqrt{\frac{EJ}{\rho Al^4}}t = ct \qquad \widetilde{x} = \frac{s}{l} \qquad \widetilde{w} = \frac{w}{l} \\
&\widetilde{\mu}_\gamma = \mu_\gamma\sqrt{\Big(\frac{EJ}{\rho Al^4}\Big)^\gamma} = \mu_\gamma c^\gamma \qquad \widetilde{q} = \frac{ql^3}{EJ} \qquad \alpha = \frac{m_p}{\rho Al} \\
&\beta = \frac{I_C}{\rho Al^3} \qquad z = kl \qquad \eta = \frac{e}{l}
\end{aligned} \tag{2.16}$$

and substituting them into Eq. (2.13), the following dimensionless equation of motion is obtained

$$\begin{aligned}
&\frac{\partial^2\widetilde{w}}{\partial\tau^2} + \frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4} + \frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\Big(\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\Big)^2 + 6\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\frac{\partial\widetilde{w}}{\partial\widetilde{x}} + 3\Big(\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\Big)^3 + \widetilde{\mu}_\gamma\left\{\frac{d^\gamma}{d\tau^\gamma}\left[\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\left(1+\frac{1}{2}\Big(\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\Big)^2\right)\right.\right. \\
&\quad \left.\left. + 3\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\frac{\partial\widetilde{w}}{\partial\widetilde{x}} + \Big(\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\Big)^2\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\right]\left(1+\frac{1}{2}\Big(\frac{\partial\widetilde{w}'}{\partial\widetilde{x}}\Big)^2\right)\right\} \\
&\quad + \widetilde{\mu}_\gamma\left\{\frac{d^\gamma}{d\tau^\gamma}\left[\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\left(1+\frac{1}{2}\Big(\frac{\partial\widetilde{w}'}{\partial\widetilde{x}}\Big)^2\right) + \Big(\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\Big)^2\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\right]\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\right\} = \widetilde{q}
\end{aligned} \tag{2.17}$$

with dimensionless boundary conditions for $\widetilde{x} = 0$, $\widetilde{w} = \widetilde{w}' = 0$ and for $\widetilde{x} = 1$

$$\begin{aligned}
&- \alpha\Big(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau^2} + \eta\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\tau^2\partial\widetilde{x}}\Big) + \frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3} + \frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3}\Big(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\Big)^2 + \frac{\partial\widetilde{w}}{\partial\widetilde{x}}\Big(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}\Big)^2 \\
&\quad + \widetilde{\mu}_\gamma\left\{\frac{d^\gamma}{dt^\gamma}\left[\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3}\left(1+\frac{1}{2}\Big(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\Big)^2\right)\right]\left(1+\frac{1}{2}\Big(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\Big)^2\right)\right\} \\
&\quad - \widetilde{\mu}_\gamma\left\{\frac{d^\gamma}{dt^\gamma}\left[\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}\left(1+\frac{1}{2}\Big(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\Big)^2\right)\right]\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right\} = 0 \\
&- (\alpha\eta^2 + \beta)\left[\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau^2}\left(1+\Big(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\Big)^2\right) + 2\Big(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau\partial\widetilde{x}}\Big)^2\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\right] \\
&\quad - \left[\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2} + \frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}\Big(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\Big)^2\right] \\
&\quad - \mu_\gamma\left\{\frac{d^\gamma}{dt^\gamma}\left[\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}\left(1+\frac{1}{2}\Big(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\Big)^2\right)\right]\left(1+\frac{1}{2}\Big(\frac{\partial\widetilde{w}(1,\tau)}{\partial\widetilde{x}}\Big)^2\right)\right\} = 0
\end{aligned} \tag{2.18}$$

The approximate solution is assumed in the form of the first eigenfunction of linearized nonlinear equation Eq. (2.17). This assumption can be made because the dynamic behavior of the analyzed beam is studied in the vicinity of the first resonance, and the applied load causes only first mode vibrations. Moreover, from the structure of Green's functions of characteristic equations (see Freundlich, 2019; Podlubny, 1999) and the fact that the first natural frequency is several times lower then the second natural frequency of the analyzed cantilever beam, it follows that the first eigenfunction has the greatest impact on the vibration amplitude. Therefore, in the first step, the equation of motion is simplified, namely, only expressions up to the third order are considered. Grouping linear and nonlinear terms, we obtain

$$\begin{aligned}&\frac{\partial^2\widetilde{w}}{\partial\tau^2}+\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}+\widetilde{\mu}_\gamma\frac{d^\gamma}{d\tau^\gamma}\Big(\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\Big)+\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\Big(\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\Big)^2+6\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\frac{\partial\widetilde{w}}{\partial\widetilde{x}}+3\Big(\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\Big)^3\\&\quad+\widetilde{\mu}_\gamma\Big\{\frac{d^\gamma}{d\tau^\gamma}\Big[\frac{1}{2}\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\Big(\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\Big)^2+3\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\frac{\partial\widetilde{w}}{\partial\widetilde{x}}+\Big(\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\Big)^3\Big]+\frac{d^\gamma}{d\tau^\gamma}\Big(\frac{\partial^3\widetilde{w}}{\partial\widetilde{x}^3}\Big)\frac{\partial\widetilde{w}}{\partial\widetilde{x}}\frac{\partial^2\widetilde{w}}{\partial\widetilde{x}^2}\Big\}=\widetilde{q}\end{aligned}\tag{2.19}$$

Using the first mode approximation, namely $\widetilde{w}(\widetilde{x},\tau)=W_1(\widetilde{x})\xi_1(\tau)$

$$\begin{aligned}&W_1D^{(2)}(\xi_1)+W_1''''\xi_1+\widetilde{\mu}_\gamma W_1''''D^{(\gamma)}(\xi_1)+W_1''''\xi_1(W_1'\xi_1)^2+6W_1''\xi_1W_1'''\xi_1W_1'\xi_1\\&\quad+3(W_1''\xi_1)^3+\widetilde{\mu}_\gamma\Big[D^{(\gamma)}\Big(\frac{1}{2}W_1''''\xi_1(W_1'\xi_1)^2+3W_1''\xi_1W_1'''\xi_1W_1'\xi_1+(W_1''\xi_1)^3\Big)\\&\quad+D^{(\gamma)}(W_1'''\xi_1)W_1'\xi_1W_1''\xi_1\Big]=\widetilde{q}\end{aligned}\tag{2.20}$$

where W_1 is the first eigenfunction of linearized nonlinear equation (2.17).

Next, multiplying both sides by W_1, substituting the relationship $W''''(\widetilde{x})=\widehat{k}^4W(\widetilde{x})$ (Eq. (2.27)), and integrating from 0 to 1, we obtain an equation of the generalized coordinate

$$D^{(2)}(\xi_1)+\widehat{k}^4\xi_1+\widetilde{\mu}_\gamma\widehat{k}^4D^{(\gamma)}(\xi_1)+a_3\xi_1^3+b_3\widetilde{\mu}_\gamma D^{(\gamma)}(\xi_1^3)+b_2\widetilde{\mu}_\gamma D^{(\gamma)}(\xi_1)\xi_1^2=Q\tag{2.21}$$

where

$$\begin{aligned}a_3&=\frac{\int_0^1\widehat{k}^4W_1^2(W_1')^2\,d\widetilde{x}+6\int_0^1W_1''W_1'''W_1'W_1\,d\widetilde{x}+3\int_0^1(W_1'')^3W_1\,d\widetilde{x}}{\int_0^1W_1^2\,d\widetilde{x}}\\b_2&=\frac{\int_0^1W_1'''W_1'W_1''W_1\,d\widetilde{x}}{\inf_0^1W_1^2\,d\widetilde{x}}\qquad Q=\frac{\int_0^1\widetilde{q}(\widetilde{x},\tau)W_1\,d\widetilde{x}}{\int_0^1W_1^2\,d\widetilde{x}}\\b_3&=\frac{\int_0^1\frac{1}{2}\widehat{k}^4W_1^2(W_1')^2\,d\widetilde{x}+3\int_0^1W_1''W_1'''W_1'W_1\,d\widetilde{x}+\int_0^1(W_1'')^3W_1\,d\widetilde{x}}{\int_0^1W_1^2\,d\widetilde{x}}\end{aligned}\tag{2.22}$$

As was mentioned earlier, the approximate solution to Eq. (2.20) is in the form of linear modes of linearized Eq. (2.19). This linearized equation has a form

$$\frac{\partial^2\widetilde{w}}{\partial\tau^2}+\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}+\mu_\gamma\frac{d^\gamma}{d\tau^\gamma}\Big(\frac{\partial^4\widetilde{w}}{\partial\widetilde{x}^4}\Big)=\widetilde{q}\tag{2.23}$$

with linearized boundary conditions, namely, for the clamped beam end $\widetilde{x}=0$

$$\widetilde{w}(0,\tau)=\frac{\partial\widetilde{w}(0,\tau)}{\partial\widetilde{x}}=0\tag{2.24}$$

and for $\tilde{x}=1$

$$\begin{aligned}&\alpha\Big(\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau^2}+\eta\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\tau^2\partial\widetilde{x}}\Big)-\Big(\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3}+\widetilde{\mu}_\gamma\frac{d^\gamma}{d\tau^\gamma}\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\widetilde{x}^3}\Big)=0\\&\alpha\eta\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\tau^2}+(\alpha\eta^2+\beta)\frac{\partial^3\widetilde{w}(1,\tau)}{\partial\tau^2\partial\widetilde{x}}+\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}+\widetilde{\mu}_\gamma\frac{d^\gamma}{d\tau^\gamma}\frac{\partial^2\widetilde{w}(1,\tau)}{\partial\widetilde{x}^2}=0\end{aligned}\tag{2.25}$$

The solution to the problem formulated by Eqs. (2.23)-(2.25) is sought in the form of a convergent series of the dimensionless beam eigenfunctions

$$\widetilde{w}(\widetilde{x},\tau) = \sum_{n=1}^{\infty} \xi_n(\tau) W_n(\widetilde{x}) \tag{2.26}$$

where $W_n(\widetilde{x})$ is the n-th eigenfunction of the beam, $\xi_n(\tau)$ is the n-th time depending generalized coordinate (Meirovitch, 1967).

The functions $W_n(\widetilde{x})$ can be determined with the help of a well-known procedure, i.e. by solving Eq. (2.23) with its right-hand side equal to zero (homogeneous equation). Namely, utilizing separation of variables, the subsequent equation may be obtained

$$W''''(\widetilde{x}) - \widehat{k}^4 W(\widetilde{x}) = 0 \tag{2.27}$$

The solution to equation above (2.27) is sought as

$$W(\widetilde{x}) = A\sin(\widehat{k}\widetilde{x}) + B\cos(\widehat{k}\widetilde{x}) + C\sinh(\widehat{k}\widetilde{x}) + D\cosh(\widehat{k}\widetilde{x}) \tag{2.28}$$

where A, B, C, D are arbitrary unknown constants.

Using the first two boundary conditions Eq. (2.24), the following relations between the constants may be found, namely, $A = -C$ and $B = -D$. Then, using these relationships and the next two boundary conditions Eq. (2.25), the following system of equations for constants A and B is derived

$$\begin{aligned} &A[\alpha\widehat{k}(\sin\widehat{k} - \sinh\widehat{k}) + \alpha\widehat{k}^2\eta(\cos\widehat{k} - \cosh\widehat{k}) - (\cos\widehat{k} + \cosh\widehat{k})] \\ &\qquad + B[\alpha\widehat{k}(\cos\widehat{k} - \cosh\widehat{k}) - \alpha\widehat{k}^2\eta(\sin\widehat{k} + \sinh\widehat{k}) - (\sin\widehat{k} - \sinh\widehat{k})] = 0 \\ &A[\alpha\eta\widehat{k}^2(\sin\widehat{k} - \sinh\widehat{k}) + (\alpha\eta^2 + \beta)\widehat{k}^3(\cos\widehat{k} - \cosh\widehat{k}) + \sin\widehat{k} + \sinh\widehat{k}] \\ &\qquad + B[\alpha\eta\widehat{k}^2(\cos\widehat{k} - \cosh\widehat{k}) - (\alpha\eta^2 + \beta)\widehat{k}^3(\sin\widehat{k} + \sinh\widehat{k}) + \cos\widehat{k} + \cosh\widehat{k}] = 0 \end{aligned} \tag{2.29}$$

The system of equations presented above, Eq. (2.29), is satisfied if the determinant of the coefficients matrix of the system of equation equals zero. Then, equating the determinant of (2.29) to zero, after long and arduous mathematical transformations, the characteristic equation of the system can be obtained

$$\begin{aligned} &-\widehat{k}^4\alpha\beta(1 - \cos\widehat{k}\cosh\widehat{k}) + \widehat{k}^3(\beta + \alpha\eta^2)(\cos\widehat{k}\sinh\widehat{k} + \sin\widehat{k}\cosh\widehat{k}) \\ &\qquad + 2\alpha\eta\widehat{k}^2\sin\widehat{k}\sinh\widehat{k} - \alpha(\cos\widehat{k}\cosh\widehat{k} - \sin\widehat{k}\cosh\widehat{k}) - \cos\widehat{k}\cosh\widehat{k} - 1 = 0 \end{aligned} \tag{2.30}$$

Characteristic equation (2.30) has of a countable infinitive set of roots $\widehat{k}_n$ corresponding to the n-th natural undamped dimensionless frequency of the beam. Next, substituting the derived roots into Eqs. (2.27) and (2.29), the expression for eigenfunctions can be obtained

$$W_n(\widetilde{x}) = A_n[\sin(\widehat{k}_n\widetilde{x}) - \sinh(\widehat{k}_n\widetilde{x}) - \lambda_n(\cos(\widehat{k}_n\widetilde{x}) - \cosh(\widehat{k}_n\widetilde{x}))] \tag{2.31}$$

where

$$\lambda_n = \frac{\alpha\widehat{k}_n(\sin\widehat{k}_n - \sinh\widehat{k}_n) + \alpha\widehat{k}_n^2\eta(\cos\widehat{k}_n - \cosh\widehat{k}_n) - (\cos\widehat{k}_n + \cosh\widehat{k}_n)}{\alpha\widehat{k}_n(\cos\widehat{k}_n - \cosh\widehat{k}_n) - \alpha\widehat{k}_n^2\eta(\sin\widehat{k}_n + \sinh\widehat{k}_n) - (\sin\widehat{k}_n - \sinh\widehat{k}_n)}$$

Eigenfunctions (2.31) must satisfy the orthogonality condition. Using the well-known standard procedure (see e.g. Meirovitch, 1967), it can be shown that the orthogonality condition has a following form

$$\begin{aligned} &\int_0^1 W_m(\widetilde{x})W_n(\widetilde{x})\,d\widetilde{x} + \alpha[W_n(1)W_m(1) + \eta W_n'(1)W_m(1)] \\ &\qquad + \alpha\eta W_n(1)W_m'(1) + (\alpha\eta^2 + \beta)W_n'(1)W_m'(1) = \delta_{mn} \end{aligned} \tag{2.32}$$

Employing orthogonality condition Eq. (2.31) and expression for eigenfunction, Eq. (2.31), coefficients A_n in Eq. (2.31) can be calculated as

$$A_n = \frac{1}{\sqrt{\int_0^1 \widetilde{W}_n^2(\widetilde{x});d\widetilde{x} + \alpha[\widetilde{W}_n^2(1) + 2\eta\widetilde{W}_n'(1)\widetilde{W}_n(1)] + (\alpha\eta^2+\beta)\widetilde{W}_n'^2(1)}} \tag{2.33}$$

Therefore, the function W_1 in Eqs. (2.20) and (2.22) is determined, thus the approximate solution to the problem described by Eq. (2.19) may be obtained.

Fractional differential equation (2.21) can be solved numerically using a method similar to the method presented in the book by Diethelm (2010). In this method, the fractional differential equation is converted to a system of mixed ordinary and fractional differential equations, each of the order $0 < \gamma \leqslant 1$. The converted system of equations contains integer and fractional order differential equations, which can be partitioned into two separated systems of equations and solved simultaneously (Freundlich, 2021). The system of equations is solved using own author's procedure implemented in the Matlab package. The fractional order differential equations are integrated using the trapezoidal rule for the fractional Caputo derivative worked out by Diethelm *et al.* (2005), while the integer order equations are integrated using the Adams--Bashforth-Moulton predictor-corrector method (see e.g Chapra and Canale, 2010). Roots of the nonlinear characteristic equation of system (2.30) are computed using Matlab procedure "fzero". The knowledge of damped natural frequencies are useful in dynamic analysis of the system. The natural damped frequencies of linearized sytem (2.23) can be calculated solving the characteristic equation associated with linearized fractional differential equation (2.21) with the zero right hand side, namely

$$s_n^2 + \widetilde{\mu}_\gamma \widehat{k}^4 s_n^\gamma + \widehat{k}^4 s = 0 \tag{2.34}$$

Characteristic equation (2.34) has two conjugate complex roots located in the left half-plane of the complex domain (Rossikhin and Shitikova, 1997). The absolute value of the real part of the root is the damping coefficient, whereas the imaginary part of the root is the natural damped frequency (Rossikhin and Shitikova, 1997). Equation (2.34) is solved using author's own procedure based on Newton's method of solving nonlinear complex equations (Chapra and Canale, 2010).

3. Example of numerical calculations and discussion

To demonstrate the usefulness of the method presented in the previous Section, examplary calculations of transient vibrations of the analyzed beam have been performed. The relationships obtained in the preceding Section are used to study the impact of the fractional derivative order and other parameters of the system on its transient vibrations. Additionally, responses of linear and nonlinear systems are studied and compared. Since it is important to know the modal damping and damped natural frequencies in the analysis of system dynamics, the effect of the order of the fractional derivative on the damping coefficient and damped natural frequency of the system is first determined. As mentioned previously, the damping coefficient and natural damped frequency of the system are determined by real and imaginary parts of the roots of Eq. (2.34), respectively. Numerical calculations are performed for various orders of the fractional derivative and for beam parameters $\alpha = 1$, $\beta = 0.005$, $\eta = 0.05$ and for two damping coefficients, $\widetilde{\mu}_\gamma = 0.008$ and 0.016. Computed relationships between the calculated roots and the order of the fractional derivative are shown in Figs. 3 and 4.

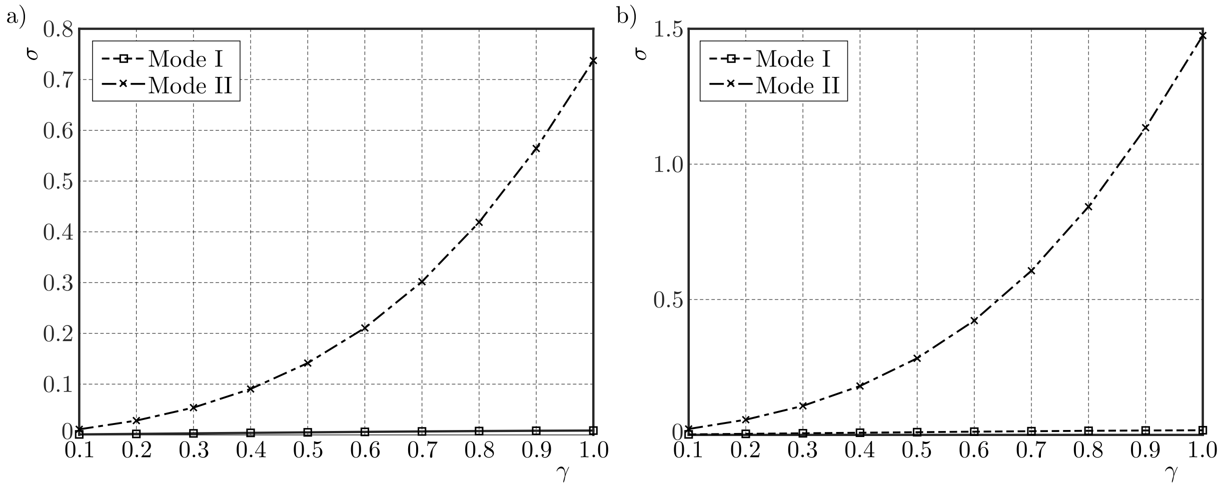

Fig. 3. The real part of the roots of characteristic equation (2.34), $\alpha = 1$, $\beta = 0.005$, $\eta = 0.05$: (a) $\widetilde{\mu}_\gamma = 0.008$, (b) $\widetilde{\mu}_\gamma = 0.016$

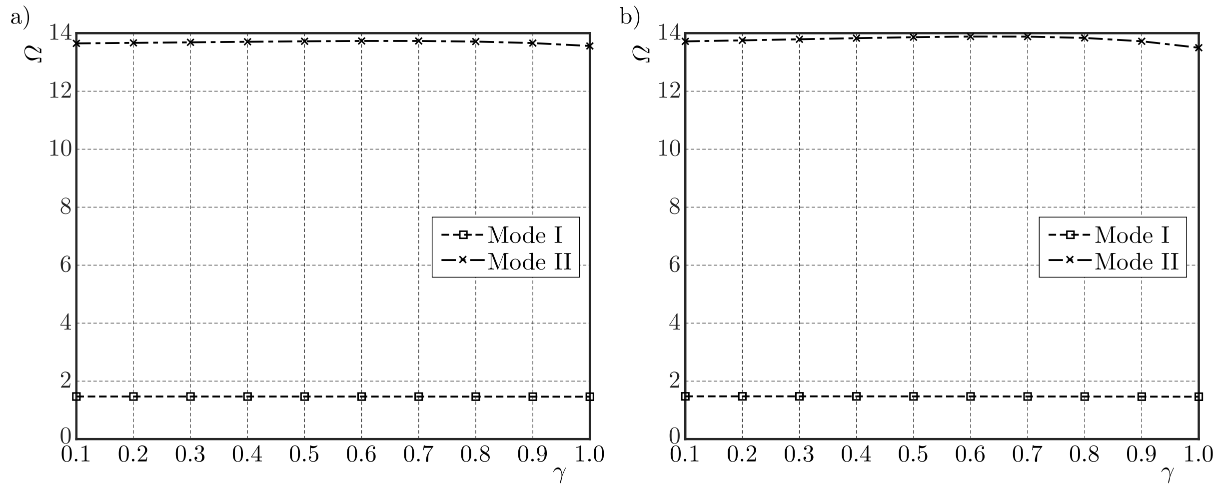

Fig. 4. The imaginary part of the roots of characteristic equation (2.34), $\alpha = 1$, $\beta = 0.005$, $\eta = 0.05$: (a) $\widetilde{\mu}_\gamma = 0.008$, (b) $\widetilde{\mu}_\gamma = 0.016$

It can be noticed from Fig. 3 that the damping coefficient exponentially increases with an increase of the order of the fractional derivative. The increase is significantly greater for the second mode of vibration. On the contrary, the damped natural frequency practically does not depend on the change of the order of the fractional derivative (see Fig. 4).

As noted before, in some vibration studies of the beam with attached at its end a heavy mass element, it is necessary to take into account the eccentricity. Therefore, sample calculations showing the effect of eccentricity on the damping coefficient and natural damped frequency are made. The calculations are made for various η coefficients, for $\gamma = 0.5$, $\alpha = 1$, $\beta = 0.005$, and for two damping coefficients, $\widetilde{\mu}_\gamma = 0.008$ and 0.016. An impact of the eccentricity coefficient η on the damping coefficient and damped natural frequency is shown in Figs. 5 and 6.

As can be seen from Figs. 4 and 6, an increase in the order of the fractional derivative results in a decrease of damping coefficients and natural damped frequencies. The decrease is greater for the second mode of vibrations. A relative difference between damped natural frequencies for $\eta = 0$ and $\eta = 0.2$ is about 20% for the fist mode of vibration, and about 24% for the second mode of vibration.

Next, having determined damped natural frequencies of the linearized system, the impact of the order of the fractional derivative on transient forced vibrations of the analyzed beam is

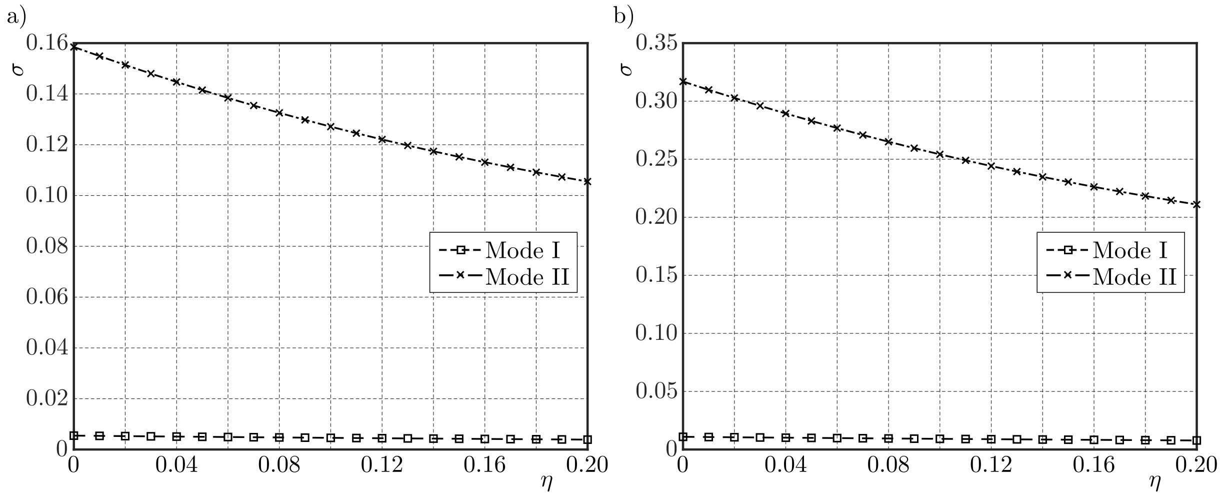

Fig. 5. The real part of the roots of characteristic equation (2.34), $\gamma = 0.5$, $\alpha = 1$, $\beta = 0.005$: (a) $\widetilde{\mu}_\gamma = 0.008$, (b) $\widetilde{\mu}_\gamma = 0.016$

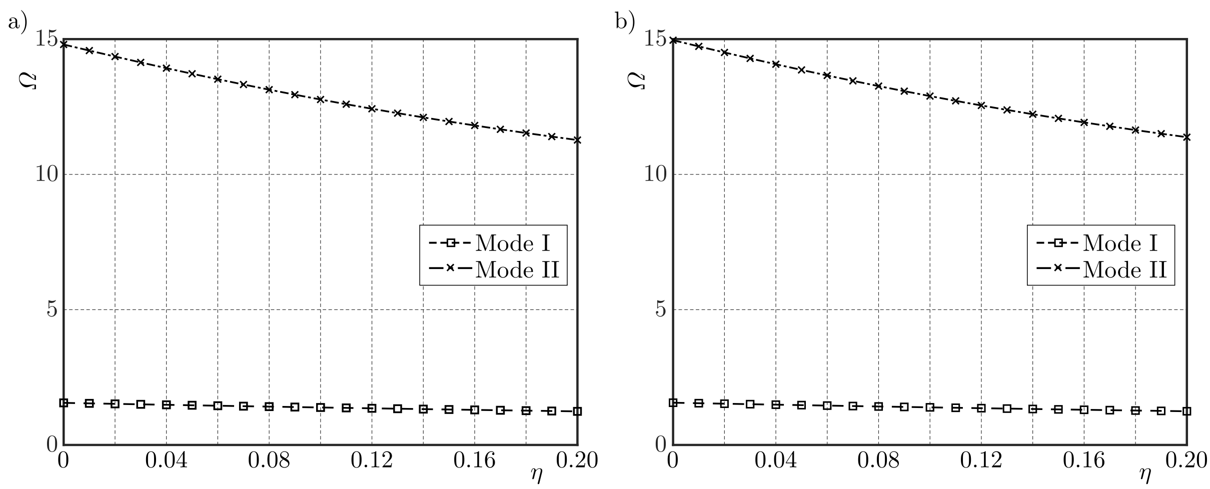

Fig. 6. The imaginary part of the roots of characteristic equation (2.34), $\gamma = 0.5$, $\alpha = 1$, $\beta = 0.005$: (a) $\widetilde{\mu}_\gamma = 0.008$, (b) $\widetilde{\mu}_\gamma = 0.016$

investigated. Linear and nonlinear transient vibrations are examined. In the first stage, linear and nonlinear beam responses to the harmonic excitation of amplitude F_0 are computed. The excitation frequency is assumed to be the natural damped frequency of the linearized system determined earlier (see Fig.4). These calculations are performed for the dimensionless beam parameters $\alpha = 1$, $\beta = 0.005$, $\eta = 0.05$, two damping coefficients, $\widetilde{\mu}_\gamma = 0.008$ and 0.016, and various orders of the fractional derivative $\gamma = 0.25, 0.5, 0.75, 1.0$. The calculated responses of the linearized system to the harmonic excitation are shown in Fig. 7, whereas for the nonlinear system are shown in Fig. 8. As can be seen from Fig. 7, the maximum amplitudes of the linearized responses increase monotonically until their values stabilize. Furthermore, vibration amplitudes are greater for lower values of the order of the fractional derivative γ for both coefficients $\widetilde{\mu}_\gamma$ (Fig. 7). In contrast, the maximum amplitudes of the nonlinear responses oscillate until their values stabilize (see Fig. 8). Furthermore, it can be seen from Fig. 8 that the oscilations of the maximum amplidudes of nonlinear responses are greater for lower values of the order of the fractional derivative γ. Similarly, as in the case of linear responses, the vibration amplitudes are lower as the order of the fractional derivative γ increases.

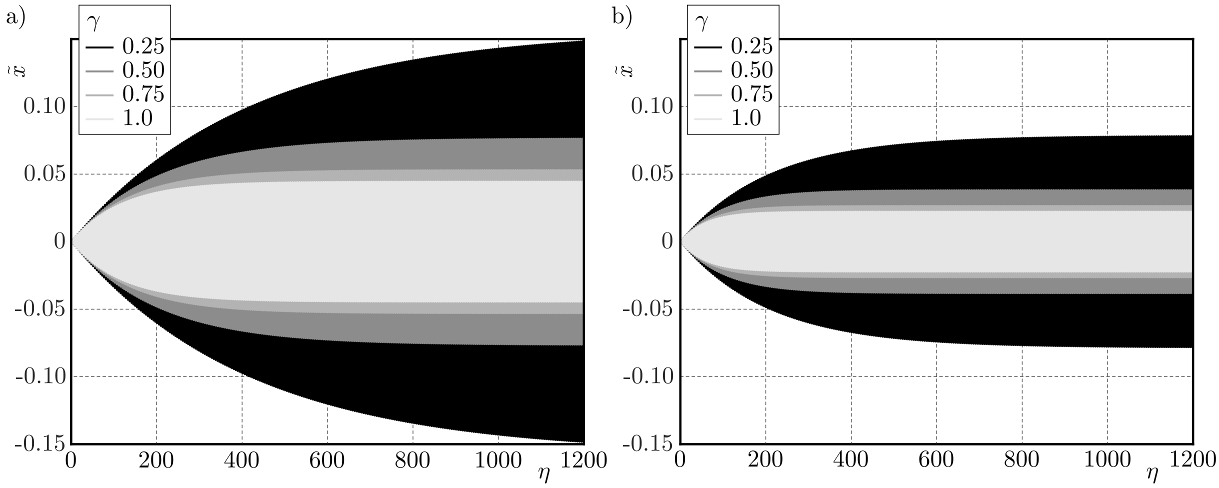

Fig. 7. Linear beam response, harmonic excitation: (a) $\widetilde{\mu}_\gamma = 0.008$, (b) $\widetilde{\mu}_\gamma = 0.016$

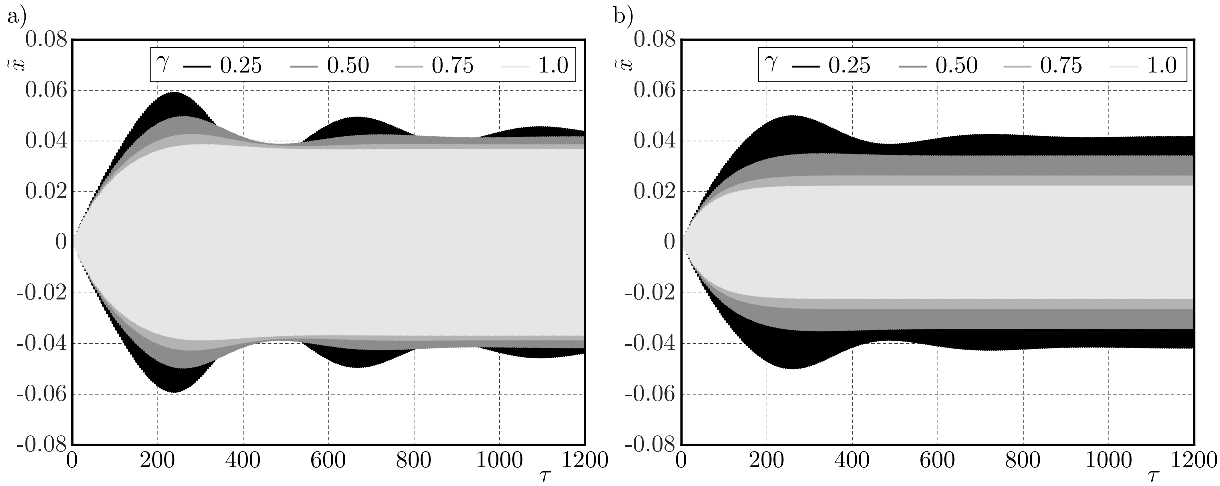

Fig. 8. Nonlinear beam response, harmonic excitation: (a) $\widetilde{\mu}_\gamma = 0.008$, (b) $\widetilde{\mu}_\gamma = 0.016$

In the next step, the transient responses of the beam to an excitation force of varying angular frequency are studied. The excitation force function is described by the following expression

$$F(\tau) = F_0 \sin \frac{\mathcal{E}\tau^2}{2} \tag{3.1}$$

where $\mathcal{E}$ is dimensionless angular acceleration.

The beam responses to excitation described by Eq. (3.1) are computed for the dimensionless angular acceleration $\mathcal{E} = 0.1$, dimensionless beam parameters $\gamma = 0.5$, $\alpha = 1$, $\beta = 0.005$, $\eta = 0.05$, two damping coefficients, $\widetilde{\mu}_\gamma = 0.008$ and 0.016, and the order of the fractional derivative $\gamma = 0.25$, 0.5, 0.75, 1.0. The calculated responses are shown in Fig. 9. The obtained responses of the beam show that the maximum amplitudes of vibrations, after reaching the maximum value, decrease monotonically. The decrease is faster for higher orders of the fractional derivative γ and greater coefficient $\widetilde{\mu}_\gamma$. As can be seen from Fig. 9, an increase of the order of the fractional derivative decreases the response amplitudes.

Analyzing the results shown in Figs. 7-9, it can be concluded that the order of the fractional derivative has a similar effect on vibration amplitudes as the damping coefficient or the time constant μ_γ, i.e., increasing the order of the fractional derivative γ causes a decrease in the vibration amplitudes.

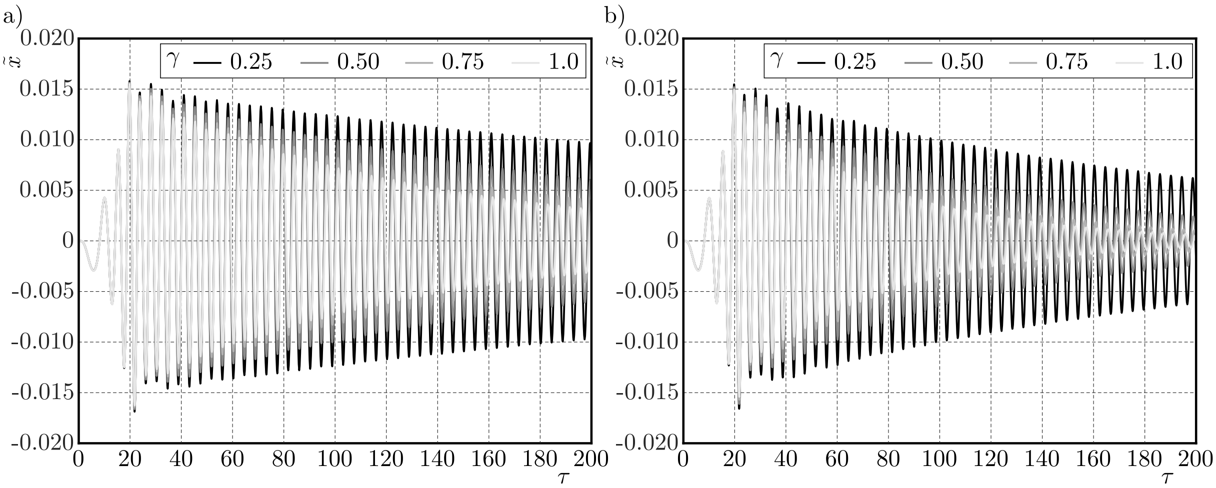

Fig. 9. Nonlinear beam response, transient excitation, $\mathcal{E} = 0.1$; (a) $\widetilde{\mu}_\gamma = 0.008$, (b) $\widetilde{\mu}_\gamma = 0.016$

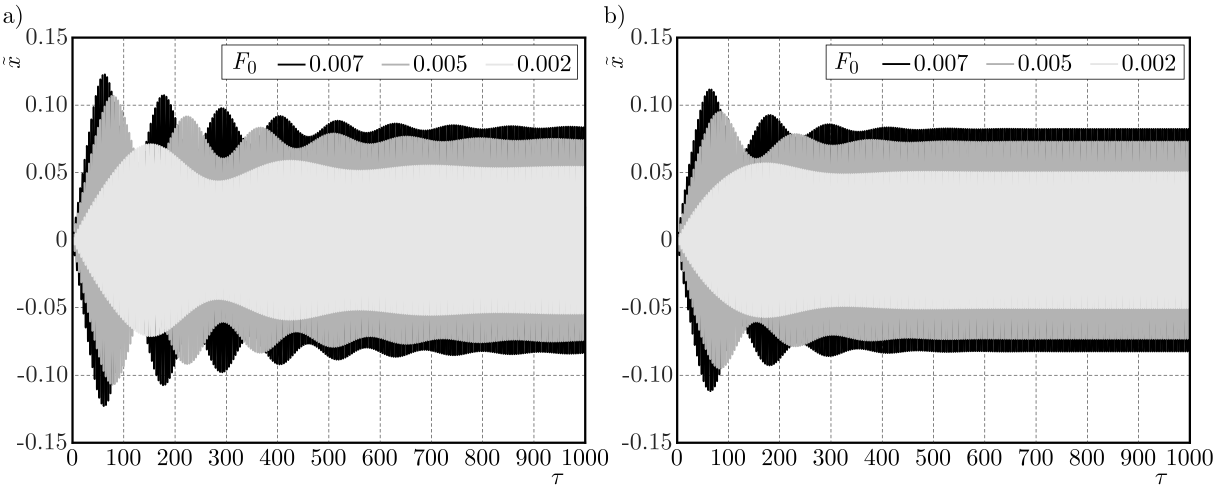

Fig. 10. Nonlinear beam response, harmonic excitation, $\gamma = 0.5$: (a) $\widetilde{\mu}_\gamma = 0.008$, (b) $\widetilde{\mu}_\gamma = 0.016$

Finally, the effect of amplitude F_0 of the sinusoidal forcing force on the transient responses of the beam is studied. The study is carried out for the order of the fractional derivative $\gamma = 0.5$, $\widetilde{\mu}_\gamma = 0.008$, 0.016, and amplitudes of the exciting force $F_0 = 0.002$, 0.005, 0.007. The computed nonlinear responses are presented in Fig. 10.

From Fig. 10 we can see that the oscillation of the maximum amplitude of the response is higher for higher amplitudes of the exciting force. Additionally, the responses reach the steady--state amplitudes after a longer time period for higher forcing amplitudes F_0.

4. Conclusions

In this paper, linear and nonlinear transient vibrations of a fractional cantilever beam with an attached eccentric mass element are presented. The fractional Kelvin-Voigt viscoelastic material model is assumed as the beam material. Nonlinear and linear equations of motion of the beam are derived using Hamilton's principle. The characteristic equation, modal frequencies, eigenfunction and orthogonality condition are obtained for linear beam vibrations. The achieved equations of motion are solved numerically. Numerical calculations are carried out for selected beam parameters. Transient responses of the beam to the harmonic and linearly time-varying increasing frequency of a sinusoidal excitation are calculated. The beam responses to the harmonic excitation are calculated for linear and nonlinear equations of motion. Comparing the determined linear and nonlinear responses, it can be seen that the maximum amplitudes of the linear responses increase monotonically until their values stabilize, whereas the maximum amplitudes of the nonlinear responses oscillate until their values stabilize.

The obtained nonlinear responses to time-varying frequency of the sinusoidal excitation reveal that the maximum vibration amplitudes decrease monotonically after reaching their maximum value. The decrease is faster for higher orders of the fractional derivative and greater dimensionless damping coefficients.

For all obtained results, it can be stated that the maximum amplitudes of vibrations decrease as the order of the fractional derivative increases in all performed calculations, which was expected.

The carried out researches show that he effect of eccentricity on natural frequencies is approximately linear. Thus, in my opinion, the eccentricity should be taken into account in some calculations if η is greater than 0.1.

In further investigations, actual parameters of the fractional Kelvin-Voigt model corresponding to the system analyzed should be determined by conducting appropriate experimental examinations.

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