JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
61, 4, pp. 847-856, Warsaw 2023
https://doi.org/10.15632/jtam-pl/172875

EULER-LAGRANGE EQUATIONS AND NOETHER'S THEOREM OF A MULTI-SCALE MECHANO-ELECTROPHYSIOLOGICAL COUPLING MODEL OF NEURON MEMBRANE DYNAMICS

Peng Wang
School of Civil Engineering and Architecture, University of Jinan, Jinan, Shangdong, P.R. China
e-mail: cea_wangp@ujn.edu.cn; sdpengwang@163.com

Noether's theorem is applied into a multi-scale mechano-electrophysiological coupling model of neuron membrane dynamics. The Euler-Lagrange equations in generalized coordinates of this model are deduced by the nonconservative Hamilton principle. The Noether symmetry criterion and conserved quantities based on the Lie point transformation group are given. The influence of external non-potential forces and material parameters on the forms of Noether conserved quantities is detailed discussed, which indicates that the conserved quantities are very depending on the loading rate and mechanical parameters of the membrane.

Keywords: Noether symmetry, conserved quantities, Hamilton principle, mechno-electro-physiological coupling, axon

1. Introduction

Symmetry and first integrals are two fundamental structures of ordinary differential equations (ODE), which can reduce the order of ODE and even can give solutions to ODE (Bluman and Anco, 2002). Since Noether revealed the relation between symmetry and conserved quantity, Noether's theorem (Noether, 1918) has been extended to many fields. Kosmann-Schwarzbach and Schwarzbach (2011) gave a comprehensive review of Noether's theorem, such as theorem for discrete equations in mathematics (Dorodnitsyn, 2001; Hydon and Mansfield, 2011; Peng, 2017; Mansfield *et al.* , 2019; Peng and Hydon, 2022), physics (Wang, 2011, 2012; Wang and Zhu, 2011), mechanics and engineering (Mei, 1993, 2004; Zhang and Chen, 2018; Zhang, 2022). Noether's symmetry always can refer to conserved quantities, it is also called variational symmetry (Peng and Hydon, 2022). Besides Noether's symmetry, there are Lie's symmetry (Olver, 1986; Chen *et al.*, 2005), Mei symmetry (Mei, 2000; Fang *et al.*, 2007; Wang and Xue, 2016; Luo *et al.*, 2018) and other symmetries (Wang, 2018).

Recently, a new model of an axon membrane that is a multi-scale memchano-electrophysiological coupling model (Drapaca, 2015) has been proposed to understand the propagation of an action potential. Though there are different viewpoints on origin of the action potential, this new model may bridge a simple way to compare micro-mechanical parameters with experiments directly, which may be helpful in clinic applications. However, differential equations describe those models as nonlinear and multi-scale, which is not easy to solve out. Symmetry analysis based on Lie's group is a powerful tool in reduction of nonlinear differential equations and getting exact solutions (Olver, 1986). However, to our knowledge, Noether's symmetry has not been introduced into this problem.

In this paper, we will applied Noether's theorem into this model, and give Noether's symmetry criterion and conserved quantities.

The construction of this paper is as following. In Section 2, we will generalize the model in (Drapaca, 2015) and give a generalized Lagrange equation of the axon dynamics. Because the author (Drapaca, 2015) supposes that the capacitor of the membrane is constant, and uses the classical Hodgkin-Huxley equation to replace the equations of dynamics describing the mechano--transduction of ionic channel activation and inactivation, so that the results cannot reflect the mechanical information of subcellular structure affecting the action potential, in fact returning into the voltage active ionic channels scenario again. Furthermore, the author supposed no external forces acting on the system, all the process is triggered by the input electric current. So, in this part we will modified the model to be able to study a more general case, which considerd parameters of the subcellular and non-potential forces model and suppose both mechanical factors or voltage factors that can activate ionic gate control. Then we will deduce the Euler--Lagrange equation. In Section 3, Noether's symmetry and conserved quantities of the neural dynamics are studied. The criterion of Noether's symmetry and the expression form of conserved quantities are given. In Section 4, we will specifically discuss the deduced conserved quantities on various conditions. The final Section concludes the paper.

2. The Euler-Lagrange equations of neural membrane dynamics

2.1. A review of the model

As we know, the membrane of an axon consists of a phospholipid bilayer with an embedded channel protein. The propagation of electric signals in the neuron system is by producing action potential accompanied with an ion channel open or shut. The action potential can induce deformation of the neuron membrane, whereas the inverse deformation of the neuron membrane can also induce the action potential, so it is a coupling process. Modelling the axon as

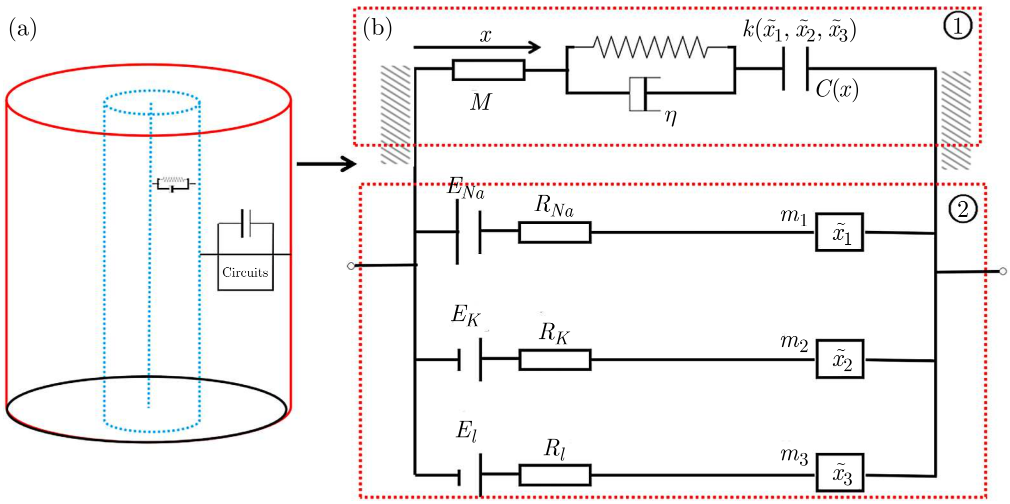

Fig. 1. The schematic of an axon (a) and mechano-electrical coupling of the axon membrane (b). The axon is an axi-symmetric homogeneous circular cylinder with intracellular space filled with axoplasm (light blue), and the outer layer is the membrane space between blue and red. By symmetry and homogeneity of the column, we study half of the axon. In cellular scale, we model the intracellular space as a viscoelastic material by the Kelvin model connects with the axon capacitor (the dotted box ①), where $(k\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ is relative to the response of cytoskeleton, where $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ is relative motion of the cytoskeleton with different ionic channels, x is displacement of the membrane. Mechanical motion or electrical stimuli can trigger the circuit. In the subcellular scale, the ionic exchange obeys the classic Hodgkin-Huxley equations (dotted box ②), but add mechanotransduction channel action that is motion of the channel protein $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$

an axi-symmetric cylinder with intracellular space and outer membrane and supposing the axon is homogenous, we can use a linear visco-elastic Kelvin-Voigt model to formulate the macro-

-mechanical process and Hodgkin-Huxley model to describe the electric process. The coupling process is unified through capacitance and membrane displacement, see Fig. 1.

The axon can be considered as an axisymmetric cylinder with a circular cross section, so we can study a half axon by symmetry. We can express the macro-mechanical kinetic energy as $T = 0.5M\dot{x}^2$, where M denotes half constant mass of the neuron of the constant cross-sectional area A and $x(t)$ is the macroscopic (cell level) displacement of the membrane depending on time, because the movement of the membrane affects the axon capacitor, which consequently induces depolarization electric current of the axon membrane. The macro mechanical potential energy is $V = -0.5kx(t)^2$. The mechanical dissipation function (work of the viscous force) $\psi_m = 0.5\eta\dot{x}^2$, where η is the viscosity coefficient. The micro relative kinetic energy of cytoskeleton structures or ionic gate control movement is $T^* = 0.5(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2)$, where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are time-dependent micro displacements of the cytoskeleton regulating activations of Na^+ and K^+ channels and, respectively, the inactivation of Na^+ channel, and varying with the deformation of the neuron membrane or conduction of the action potential, and m_1, m_2, m_3 are constant masses of mechno-sensitive channel proteins or lipid rafts. Here, the explanation T^* and components therein, is different from electrical kinetic energy in (Drapaca, 2015), but instead as the kinetic energy of the cytoskeleton from the point of view of mechanotransduction in intracellular. The electric energy of capacitor is $W_e = 0.5e_c^2/C(x)$, where $C(x)$ is capacitance of the membrane. The electric dissipation function is $\psi_e = 0.5R_{Na}\dot{e}_{Na}^2 + 0.5R_K\dot{e}_K^2 + 0.5R_l\dot{e}_l^2 + E_{Na}\dot{e}_{Na} + E_K\dot{e}_K + E_l\dot{e}_l$, where currents denote the transmembrane current induced by the membrane deformation or action potential.

2.2. The Lagrange equation of the model

Based on the conservation law of charge, we have holonomic constraint to the charge: $e_C - e_{Na} - e_K - e_l = 0$, so the number if degrees of freedom of the coupling system is seven. Introduce generalized coordinates to express universally the spatial and electrical variables q_s $(s = 1,\ldots,4,4+1,\ldots,7)$, where $q_1 = x$, $q_2 = x_1$, $q_3 = x_2$, $q_4 = x_3$ and $q_5 = e_{Na}$, $q_6 = e_K$, $q_7 = e_l$. The Lagrangian of the neuronal axon membrane mechano-electrophysiological model is

$$L(t, q_s(t)\dot{q}_s(t)) = T + T^* - V - W_e \tag{2.1}$$

The virtual work of nonconservative generalized forces is

$$\delta W(t, q_s, \dot{q}_s) = -\Big(\frac{\partial(\psi_m + \psi_e)}{\partial\dot{q}_s} - Q_s\Big)\delta q_s \tag{2.2}$$

The Hamilton principle of the nonconservative mechano-electrophysiological system of the axon membrane is

$$\int\limits_0^t (\delta L + \delta W)\,dt = 0 \tag{2.3}$$

By expanding the above equation, and using the communication relation $d\delta = \delta d$ which holds for holonomic constrained systems, and end points relations $\delta q(0) = 0$, $\delta q(t) = 0$, we can get mechano-electrophysiological coupling Euler-Lagrange equations of the axon membrane dynamics

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial\psi}{\partial\dot{q}_s} = Q_s \qquad s = 1,\ldots,7 \tag{2.4}$$

where $\psi = \psi_m + \psi_e$. The coupling equations of motion describe the changes of ions between the outer and intercellular space and micro and macro deformation of the neuron membrane.

Take the very expression of Lagrangian L, Eq. (2.1), into Eq. (2.4), then we can obtain Euler-Lagrange differential equations the same as in (Drapaca, 2015)

$$M\ddot{x} + kx + \eta\dot{x} - \frac{1}{2}\frac{\partial C}{\partial x}\Big(\frac{q_C}{C}\Big)^2 = Q_1 \tag{2.5}$$

and

$$m_1\ddot{x}_1 + \frac{1}{2}\frac{\partial k}{\partial x_1} = Q_2 \qquad m_2\ddot{x}_2 + \frac{1}{2}\frac{\partial k}{\partial x_2} = Q_3 \qquad m_3\ddot{x}_3 + \frac{1}{2}\frac{\partial k}{\partial x_3} = Q_4 \tag{2.6}$$

and

$$R_{Na}\dot{e}_{Na} + E_{Na} = V \qquad R_K\dot{e}_K + E_K = V \qquad R_l\dot{e}_l + E_l = V \tag{2.7}$$

where $V = U_i = q_C/C$ is potential of the capacitor. Kirchoff's current law demands $\dot{q}_C = \dot{q}_{Na} + \dot{q}_K + \dot{q}_l$. Take Eqs. (2.7) into Kirchoff's current law, the well-known Hodgkin-Huxley equation of the membrane potential can be found

$$\frac{d}{dt}(CV) = I - \frac{1}{R_{Na}}(V - E_{Na}) - \frac{1}{R_K}(V - E_K) - \frac{1}{R_l}(V - E_l) \tag{2.8}$$

where I is the external stimulus current.

Remark 1. *(i) In Ref. (Drapaca, 2015), the author supposed external non-potential forces* $Q_s = 0$*, that is the coupling process is triggered all by electric current which is not accordance with other supposed mechanical signals which can also induce action potential (Heimburg and Jackson, 2005), so for an alternative in the present paper, we suppose that both of them can induce the action potential.*

(ii) At the some time, Drapaca uses the Hodgkin-Huxley equation to replace equations (2.6) which makes the parameters of m_1, m_2, m_3, k, E_{Na}, E_K, E_l, R_1, R_2, R_3 *all depend on voltage* V*, which may reduce the model into Hodgkin's and Huxley's electric paradigm.*

(iii) Though the parameters m_1, m_2, m_3, k, E_{Na}, E_K, E_l, R_1, R_2, R_3 *are difficult to prescribe due to insufficient knowledge of neuronal mechanotransduction processes as Drapaca said in (Drapaca, 2015), we try to discuss their influences on the conserved quantities of the axon membrane in theory which may be useful for future experiment design.*

In the following study we will treat the general cases by Noether's symmetry analysis.

3. Noether's symmetry and conserved quantities of the neuronal membrane dynamics

We introduce a one-parameter infinitesimal Lie point transformation group in space (t, q_s)

$$t^* = t + \varepsilon\xi_0(t, \mathbf{q}) \qquad q_s^* = q_s + \varepsilon\xi_s(t, \mathbf{q}) \qquad s = 1, \dots, 7 \tag{3.1}$$

where ε is an infinitesimal parameter, $\xi_0(t, \mathbf{q})$, $\xi_s(t, \mathbf{q})$ are infinitesimal transformation generators. The infinitesimal generator vector

$$X^{(0)} = \frac{\partial}{\partial t}\xi_0(t, \mathbf{q}(t)) + \frac{\partial}{\partial q_s}\xi_s(t, \mathbf{q}(t)) \tag{3.2}$$

which is the operator for the infinitesimal generator of the one-parameter Lie group of transformations (3.1) in space $(t,\mathbf{q})$. The first prolongation of the infinitesimal generator vector is

$$X^{(1)}=X^{(0)}+\frac{\partial}{\partial\dot{q}_s}[\dot{\xi}_s(t,\mathbf{q}(t))-\dot{\xi}_0(t,\mathbf{q}(t))\dot{q}_s(t)] \tag{3.3}$$

The second prolongation of the infinitesimal generator vector is

$$X^{(2)}=X^{(1)}+\frac{\partial}{\partial\dot{q}_s}[\ddot{\xi}_s(t,\mathbf{q}(t))-2\ddot{q}_s(t)\dot{\xi}_0(t,\mathbf{q}(t))-\dot{q}_s(t)\ddot{\xi}_0(t,\mathbf{q}(t))] \tag{3.4}$$

which defines the first or second extended one-parameter Lie group of transformation in space $(t,\mathbf{q},\dot{\mathbf{q}})$ or space $(t,\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})$ by partial derivatives, where $(\dot{\bullet})$ means the first derivative to t, $(\ddot{\bullet})$ means the second derivative to t.

The Hamilton action is

$$S(\gamma)=\int\limits_{t_0}^{t_1}L(t,q_s,\dot{q}_s)\,dt \tag{3.5}$$

Under the infinitesimal transformation, the curve γ is transformed to curve γ^*. The corresponding Hamilton action is transformed to

$$S(\gamma^*)=\int\limits_{t_0^*}^{t_1^*}L(t^*,q_s^*,\dot{q}_s^*)\,dt^* \tag{3.6}$$

The variation ΔS of the Hamilton action S is the main linear part of the difference $S(\gamma^*-S(\gamma))$ to the infinitesimal parameter ε, then we have

$$\Delta S=\int\limits_{t_0}^{t_1}[\Delta L+L(\Delta t)^\bullet]\,dt \tag{3.7}$$

where Δ denotes anisochronous variation, and δ denotes isochronous variation. Expanding the above equation, we have

$$\Delta S=\int\limits_{t_0}^{t_1}\Big(L\frac{d}{dt}\Delta t+\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial\dot{q}_s}\Delta\dot{q}_s\Big)\,dt \tag{3.8}$$

Replace the infinitesimal transformation Eq. (3.1) into Eq. (3.8), and use the relation $\delta q_s=\Delta q_s-\dot{q}_s\Delta t=\varepsilon(\xi_s-\dot{q}_s\xi_0)$, $\Delta\dot{q}_s=(\Delta q_s)^\bullet-\dot{q}_s(\Delta t)^\bullet$, then the following expression can be obtained

$$\Delta S=\int\limits_{t_0}^{t_1}\Big\{\frac{d}{dt}\Big[L\xi_0+\frac{\partial L}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)\Big]+\Big(\frac{\partial L}{\partial q_s}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_s}\Big)(\xi_s-\dot{q}_s\xi_0)\Big\}\,dt \tag{3.9}$$

Definition 1. If the variation of Hamilton action satisfies

$$\Delta S=0 \tag{3.10}$$

infinitesimal transformation (3.1) is the Noether symmetrical transformation.

Based on Definition 1, we can get the Noether symmetry criterion.

Criterion 1. If the infinitesimal generators ξ_0, ξ_s satisfy

$$L\dot{\xi}_0 + X^1(L) = 0 \tag{3.11}$$

the transformation invariance is named Noether's symmetry, which is also called variational symmetry.

For Noether's symmetry, we can deduce the conserved quantities.

Theorem 1. For a Lagrangian system, if the generators $\xi_0(t, \mathbf{q})$, $\xi_s(s, \mathbf{q})$ of infinitesimal transformations is Noether's symmetry, there exist conserved quantities as

$$I_N = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = \text{const} \tag{3.12}$$

which are called Noether conserved quantities. We can directly deduce this result from Eq. (3.9).

In fact, we can generalize the Noether symmetry to non-conservative dynamical systems.

Definition 2. If the Hamilton action is a generalized quasi-invariant under an infinitesimal transformation group, that is, the variation satisfies

$$\Delta S = -\int_{t_0}^{t_1} \Big[\frac{d}{dt}(\Delta G) + \Big(Q_s - \frac{\partial \psi}{\partial \dot{q}_s}\Big)\delta q_s\Big]\, dt \tag{3.13}$$

infinitesimal transformation (3.1) is a generalized quasi-symmetrical transformation, where $G(t, \mathbf{q}, \dot{\mathbf{q}})$ is a gauge function, and $(Q_s - \partial\psi/\partial\dot{q}_s)\delta q_s$ is the sum of virtual work of generalized non-conservative forces.

Based on Definition 2, we can get the generalized Noether symmetry criterion.

Criterion 2. If there exists a gauge function $G(t, \mathbf{q}, \dot{\mathbf{q}})$ making the infinitesimal generators ξ_0, ξ_s satisfy

$$L\dot{\xi}_0 + X^1(L) + \Big(Q_s - \frac{\partial \psi}{\partial \dot{q}_s}\Big)(\xi_s - \dot{q}_s\xi_0) + \dot{G}_N = 0 \tag{3.14}$$

the infinitesimal transformation is named a quasi-Noether symmetry.

The Noether symmetry always can lead to conserved quantities.

Theorem 2. For Lagrange equation Eq. (2.4) of the neuronal membrane dynamics, if the infinitesimal generators $\xi_0(t, \mathbf{q})$, $\xi_s(s, \mathbf{q})$ satisfy Criterion 2, the system has the following first integrals

$$I_N = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_N = \text{const} \tag{3.15}$$

which are also Noether conserved quantities.

Proof: Expanding Definition 2, we have

$$\Delta S = \int_{t_0}^{t_1} \Big\{\frac{d}{dt}\Big[L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0)\Big] + \Big(\frac{\partial L}{\partial q_s} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - Q_s + \frac{\partial \psi}{\partial \dot{q}_s}\Big)(\xi_s - \dot{q}_s\xi_0)\Big\}\, dt = 0 \tag{3.16}$$

considering Eq. (2.4), we can get the results directly.

4. Noether's symmetry generators and conserved quantities

Take the exact form of Lagrangian L and dissipative function ψ into Noether identity Eq. (3.14), then we have

$$\begin{aligned}&\xi_1\Big(-kq_1+\frac{V^2}{2}\frac{\partial C(q_1)}{\partial q_1}\Big)+M\dot{q}_1(\dot{\xi_1}-\dot{q}_1\dot{\xi}_0)+(Q_1-\eta\dot{q}_1)(\xi_1-\dot{q}_1\xi_0)-\xi_2\frac{1}{2}q_1^2\frac{\partial k}{\partial q_2}\\&\quad+m_2\dot{q}_2(\dot{\xi_2}-\dot{q}_2\dot{\xi}_0)+Q_2(\xi_2-\dot{q}_2\xi_0)-\xi_3\frac{1}{2}q_1^2\frac{\partial k}{\partial q_3}+m_3\dot{q}_3(\dot{\xi_3}-\dot{q}_3\dot{\xi}_0)\\&\quad+Q_3(\xi_3-\dot{q}_3\xi_0)-\xi_4\frac{1}{2}q_1^2\frac{\partial k}{\partial q_4}+m_4\dot{q}_4(\dot{\xi_4}-\dot{q}_4\dot{\xi}_0)+Q_4(\xi_4-\dot{q}_4\xi_0)\\&\quad+(Q_5-R_{Na}\dot{q}_5-E_{Na})(\xi_5-\dot{q}_5\xi_0)-\xi_5V+(Q_6-R_K\dot{q}_6-E_K)(\xi_6-\dot{q}_6\xi_0)-\xi_6V\\&\quad+(Q_7-R_l\dot{q}_8-E_l)(\xi_7-\dot{q}_7\xi_0)-\xi_7V+L\dot{\xi}_0+\dot{G}_N=0\end{aligned}\tag{4.1}$$

Next, let us discuss the structures of Noether conserved quantities when the external nonpotential forces $Q_s \neq 0$ $(s=1,2,3,4)$. If $k=$ const and the total charges of the systems is invariant, we can get solutions

$$\xi_0=\pm1 \qquad \xi_1=\pm\dot{q}_1 \qquad \xi_2=\xi_3=\xi_4=0 \qquad \xi_i=\pm\dot{q}_s \quad (s=5,6,7)\tag{4.2}$$

$$\xi_0=\pm1 \qquad \xi_i=\pm\dot{q}_s \quad (s=1,2,3,4)\tag{4.3}$$

$$\xi_i=\pm\dot{q}_s \quad (s=5,6,7)\tag{4.4}$$

$$\xi_0=\xi_1=0 \qquad \xi_2=\xi_3=\xi_4=\pm1 \qquad \xi_5=\xi_6=\xi_7=0\tag{4.5}$$

$$\xi_0=\xi_1=0 \qquad \xi_i=\pm\dot{q}_s \quad (s=2,3,4) \qquad \xi_5=\xi_6=\xi_7=0\tag{4.6}$$

The corresponding Noether conserved quantities are

$$\begin{aligned}&I_{N11}=\pm\Big(\frac{1}{2}m_2\dot{q}_2^2+\frac{1}{2}m_3\dot{q}_3^2+\frac{1}{2}m_4\dot{q}_4^2-W_q\Big) \qquad I_{N12}=0\\&I_{N13}=\pm(m_2\dot{q}_2+m_3\dot{q}_3+m_4\dot{q}_4-Q_q) \qquad I_{N14}=-I_{N11}\end{aligned}\tag{4.7}$$

Here, the composition of conserved quantities (4.7) depends on specific non-potential forces. W_q has several forms

$$\begin{aligned}&Q_2=\ddot{q}_2 \qquad Q_3=\ddot{q}_3 \qquad Q_4=\ddot{q}_4 \qquad W_{q1}=\frac{1}{2}(\dot{q}_2^2+\dot{q}_3^2+\dot{q}_4^2)\\&Q_2=\ddot{q}_3 \qquad Q_3=\ddot{q}_2 \qquad Q_4=\ddot{q}_4=0 \qquad W_{q2}=\dot{q}_2\dot{q}_3+\frac{1}{2}\dot{q}_4^2=0\\&Q_2=\ddot{q}_4 \qquad Q_3=\ddot{q}_3=0 \qquad Q_4=\ddot{q}_2 \qquad W_{q3}=\dot{q}_2\dot{q}_4+\frac{1}{2}\dot{q}_3^2=0\\&Q_2=\ddot{q}_2=0 \qquad Q_3=\ddot{q}_4 \qquad Q_4=\ddot{q}_3 \qquad W_{q4}=\dot{q}_3\dot{q}_4+\frac{1}{2}\dot{q}_2^2=0\end{aligned}\tag{4.8}$$

and $Q_q=\dot{q}_2+\dot{q}_3+\dot{q}_4$ or a combination of $\dot{q}_s$ $(s=2,3,4)$. We point out that for solution (4.2)-(4.6), always holding $\xi_s-\dot{q}_s\xi_0=0$, the non-potential forces have no action on the Noether identities.

If $k=$ const, $C=$ const, we also have solutions (4.2)-(4.6), and the corresponding conserved quantities are

$$I_{N21}=I_{N11}\mp\frac{e_C^2}{2C} \qquad I_{N22}=-\frac{e_C^2}{2C} \qquad I_{N23}=I_{N13} \qquad I_{N24}=I_{N14}\tag{4.9}$$

We can get that for infinitesimal generators (4.5) and (4.6), the capacitance does not affect the conserved quantities.

If $k \neq$ const, $C =$ const, we have one solution (4.4), and the corresponding conserved quantities are $I_{N31} = -e_C^2/2C$. For $k \neq$ const, $C \neq$ const, we have one solution (4.4) with the trivial invariant $I_N = 0$.

There is a particular case $Q_1 = \eta \dot{q}_1$ in which the external non-potential force is synchronized with viscosity of the axon membrane material. Let us study the conserved quantities for this case. One solution of Noether's identity is

$$\xi_0 = \pm 1 \qquad \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0 \qquad \xi_s = \pm \dot{q}_s \quad (s = 5, 6, 7) \tag{4.10}$$

The corresponding Noether conserved quantities are

$$I_N = \mp\Big(\frac{1}{2}M\dot{q}_1^2 + \frac{1}{2}kq_1^2 + \frac{e_C^2}{2C} + \frac{1}{2}m_2\dot{q}_2^2 + \frac{1}{2}m_3\dot{q}_3^2 + \frac{1}{2}m_4\dot{q}_4^2\Big) \tag{4.11}$$

Furthermore, if $k =$ const and the total charge of the system is invariant, we can get solutions (4.2) and (4,4) and corresponding conserved quantities with $I_{N61} = I_{N11}$, $I_{N62} = 0$, and other two solutions

$$\begin{array}{llll} \xi_0 = \pm 1 & \xi_1 = \mp \dot{q}_1 & \xi_s = 0 \quad (s = 2, 3, 4) & \xi_s = \pm \dot{q}_s \quad (s = 5, 6, 7) \\ \xi_0 = \pm 1 & \xi_1 = \mp \dot{q}_1 & \xi_s = 0 \quad (s = 2, 3, 4) & \xi_s = \pm \dot{q}_s \quad (s = 5, 6, 7) \end{array} \tag{4.12}$$

The corresponding Noether conserved quantities are

$$\begin{aligned} I_{N63} &= \mp\Big(M\dot{q}_1^2 + kq_1^2 + \frac{e_C^2}{C} + \frac{1}{2}m_2\dot{q}_2^2 + \frac{1}{2}m_3\dot{q}_3^2 + \frac{1}{2}m_4\dot{q}_4^2 - W_q\Big) \\ I_{N64} &= \pm\Big(\frac{1}{2}M\dot{q}_1^2 + \frac{1}{2}kq_1^2 + \frac{e_C^2}{2C}\Big) \end{aligned} \tag{4.13}$$

If $k =$ const, $C =$ const, we also have solutions (4.2)-(4.6) and (4.12), and the corresponding conserved quantities are $I_{N71} = I_{N21}$, $I_{N72} = I_{N22}$, $I_{N73} = I_{N13}$, $I_{N74} = I_{N14}$, and

$$\begin{aligned} I_{N75} &= \mp\Big(M\dot{q}_1^2 + kq_1^2 + \frac{e_C^2}{2C} + \frac{1}{2}m_2\dot{q}_2^2 + \frac{1}{2}m_3\dot{q}_3^2 + \frac{1}{2}m_4\dot{q}_4^2 - W_q\Big) \\ I_{N76} &= \pm\Big(\frac{1}{2}M\dot{q}_1^2 + \frac{1}{2}kq_1^2\Big) \end{aligned} \tag{4.14}$$

and other solutions and conserved quantities, for example,

$$\begin{array}{llll} \xi_0 = 1 & \xi_1 = -\dot{q}_1 & \xi_s = \dot{q}_s \quad (s = 2, 3, 4, 5, 6, 7) & \\ \xi_0 = 1 & \xi_s = -\dot{q}_s \quad (s = 1, 2, 3, 4) & \xi_s = \dot{q}_s \quad (s = 5, 6, 7) & \end{array} \tag{4.15}$$

The corresponding Noether conserved quantities are

$$\begin{aligned} I_{N77} &= -M\dot{q}_1^2 - kq_1^2 - \frac{e_C^2}{2C} \\ I_{N78} &= I_{N77} - m_2\dot{q}_2^2 - m_3\dot{q}_3^2 - m_4\dot{q}_4^2 + 2W_q \end{aligned} \tag{4.16}$$

In fact in (4.15), the generators ξ_s $(s = 2, 3, 4)$ have a few combination types.

If $k \neq$ const, $C =$ const, we have one solution (4.4), and the corresponding conserved quantities are $I_N = 0$. For $k \neq$ const, $C \neq$ const, we have one solution (4.4) with the trivial invariant $I_N = -e_C^2/2C$. For only $k =$ const, we have solutions (4.5) and (4.6) with corresponding conserved quantities as

$$\begin{aligned} I_{N81} &= -\frac{1}{2}\Big(M\dot{q}_1^2 + kq_1^2 + \frac{e_C^2}{2C} + m_2\dot{q}_2(\dot{q}_2 - 1) + m_3\dot{q}_3(\dot{q}_3 - 1) + m_4\dot{q}_4(\dot{q}_4 - 1) - W_q + Q_q\Big) \\ I_{N82} &= I_{N14} \end{aligned} \tag{4.17}$$

In this Section, we have discussed the effects of parameters k, C and non-potential forces Q_s on the forms of Noether conserved quantities.

Remark 2. *From the above calculation we can conclude that the Noether symmetry and Noether conserved quantities are strongly determined by non-potential forces and material parameters.*

5. Conclusion

Noether's theorem is applied in a multi-scale mechano-electrophysiological model of an axon membrane. Euler-Lagrange equations of the mechano-electrophysiological model of the neuron membrane are given through which one can deduce the classical H-H equation. Noether's symmetry criterion and Noether's conserved quantities are given under the Lie point transformations group. Through Noether criterion, we work out some solutions and give out the corresponding Noether's conserved quantities under different external stimuli. During calculation, we discovered that the Noether symmetry and Noether conserved quantities are strongly determined by non-potential forces and material parameters, which may be useful for an experiment design. As solving Noether's identities, we suppose that some material parameters are constants such as k, η, However the value of material parameters are difficult to determine, and they may be found by further stability analysis. As the axon membrane is an anisotropic diphasic soft material, the fractional derivative model (Drapaca, 2017) may be more suitable to describe its behavior, and we will analyze its Noether's symmetry in another paper.

Acknowledgments

This work was partly supported by National Natural Science Foundation of China Grant No. 12272148 and No. 11772141.

References

1. Bluman G.W., Anco S.C., 2002, *Symmetry and Integration Methods for Differential Equation*, Springer-Verlag, Berlin
2. Chen X.W., Li Y.M., Zhao Y.H., 2005, Lie symmetries, perturbation to symmetries and adiabatic invariants of Lagrange system, *Physics Letters A*, **337**, 274-278
3. Dorodnitsyn V., 2001, Noether-type theorems for difference equations, *Appllied Numerical Mathematics*, **39**, 307-321
4. Drapaca C.S., 2015, An electromechanical model of neuronal dynamics using Hamilton's principle, *Frontiers in Cellular Neuroscience*, **9**, 1-8
5. Drapaca C.S., 2017, Fractional calculus in neuronal electromechanics, *Journal of Mechanics of Materials and Structures*, **12**, 1, 35-55
6. Fang J.H., Ding N., Wang P., 2007, A new type of conserved quantity of Mei symmetry for Lagrange system, *Chinese Physics*, **16**, 887
7. Heimburg T., Jackson A.D., 2005, On soliton propagation in biomembranes and nerves, *Proceedings of the National Academy of Sciences*, **102**, 28, 9790-9795
8. Hydon P., Mansfield E., 2011, Extensions of Noether's second theorem: from continuous to discrete systems, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **467**, 3206-3221
9. Kosmann-Schwarzbach Y., Schwarzbach B.E., 2011, *The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century*, Springer
10. Luo S.K., Yang M.J., Zhang Y.T., Dai Y., 2018, Basic theory of fractional Mei symmetrical perturbation and its applications, *Acta Mechanica*, **229**, 1833-1848

11. Mansfield E., Rojo-Echeburúa A., Hydon P., Peng L., 2019, Moving frames and Noether's finite difference conservation laws I, *Transactions of Mathematics and its Applications*, **3**

12. Mei F.X., 1993, Noether's theory of Birkhoffian systems, *Science in China, Serie A*, **36**, 12, 1456-1467

13. Mei F.X., 2000, Form invariance of Lagrange system, *Journal of Beijing Institute of Technology*, **9**, 120-124

14. Mei F.X., 2004, *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (in Chinese), Beijing Institute of Technology Press, Beijing

15. Noether E., 1918, Invariante Variationsprobleme, *Nachr. Akad. Wiss. Göttingen, Mathematical Physic*, **2**, 235-257

16. Olver P.J., 1986, *Applications of Lie Groups to Differential Equations*, Springer, New York

17. Peng L., 2017, Symmetries, conservation laws, and Noether's theorem for differential-difference equations, *Studies in Applied Mathematics*, **139**, 457-502

18. Peng L., Hydon P., 2022, Transformations, symmetries and Noether theorems for differential-difference equations, *Proceedings of the Royal Society, A: Mathematical, Physical and Engineering Sciences*, **478**, 20210944

19. Wang P., 2011, Perturbation to Noether symmetry and Noether adiabatic invariants of discrete mechanico-electrical systems, *Chinese Physics Letters*, **28**, 040203-4

20. Wang P., 2012, Perturbation to symmetry and adiabatic invariants of discrete nonholonomic nonconservative mechanical system, *Nonlinear Dynamics*, **68**, 53-62

21. Wang P., 2018, Conformal invariance and conserved quantities of mechanical system with unilateral constraints, *Communications in Nonlinear Science and Numerical Simulation*, **59**, 463-471

22. Wang P., Xue Y., 2016, Conformal invariance of Mei symmetry and conserved quantities of Lagrange equation of thin elastic rod, *Nonlinear Dynamics*, **83**, 1815-1822

23. Wang P., Zhu H.J., 2011, Perturbation to symmetry and adiabatic invariants of general discrete holonomic dynamical systems, *Acta Physica Polonica A*, **119**, 298-303

24. Zhang Y., 2022, Nonshifted dynamics of constrained systems on time scales under Lagrange framework and its Noether's theorem, *Communications in Nonlinear Science and Numerical Simulation*, **108**, 106214

25. Zhang H.B., Chen H.B., 2018, Noether's theorem of Hamiltonian systems with generalized fractional derivative operators, *International Journal of Non-Linear Mechanics*, **107**, 34-41

Manuscript received August 29, 2022; accepted for print September 4, 2023