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ON MITIGATION OF OSCILLATIONS OF A MECHANICAL SYSTEM WITH TWO DEGREES OF FREEDOM IN THE VICINITY OF EXTERNAL RESONANCES

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In this article, we study dynamical behaviour of a 2-DOF mechanical system subjected to an external harmonic force. This system which consists of the Duffing oscillator considered as a bulk system and a linear dynamic vibration absorber (LDVA) attached to it. An analytical approach for optimal choice of the parameters of the LDVA is suggested with the aim to avoid the "superfluous" increase in the amplitude of forced oscillations of the main system. The analysis performed shows that when using a linear absorber, its proper tuning (choice of stiffness and damping coefficients) gives satisfactory results – the peak values of the frequency-amplitude curve (FAC) are decreasing comparatively with the case of the linear main oscillator.

Keywords: vibration absorber, averaging method, resonant frequency, peaks equalizing

1. Intoduction

Undesirable vibrations occur in many areas of human activity. They arise as a result of continuous operation of machines in industry, as a result of earthquakes that are transmitted to neighboring structural elements, due to motion of an object in a stream of a liquid or gas etc. Appearance of such vibrations generates serious problems which may lead to malfunction of equipment in many engineering areas including rotating machinery, aeronautics, seafaring, space-crafting and robotics. Eliminating or mitigation of vibrations is the main goal of various industrial and technical practices (Jangid, 2021; Kremer and Liu, 2017; Vakakis *et al.*, 2009; Lu *et al.*, 2018; Balaji *et al.*, 2021). One of the possible ways to counteract unwanted vibrations is the use of passive vibration absorbers. For the first time, such a device was patented by Frahm (1911), and later a mathematical model was presented and analyzed by Ormondroyd and Den Hartog (1928), Den Hartog (1934) and Brock (1946).

Initially, a vibration absorber was considered as a linear single degree of freedom (DOF) spring-mass system that abolishes or reduces excessive vibration of a harmonic-excited system. In the literature, several terms are used for such devices: dynamic vibration absorber (DVA), tuned mass damper (TMD) or inertial damper. Numerous types of non-linear absorbers are in use today (Ocak *et al.*, 2022). Among them are: pendulum-like absorbers, torsional absorbers, absorbers with quasi-zero or negative stiffness, vibro-impact dampers and many others.

Over the past two decades the efforts of many authors have been directed to the study of dynamics of a 2-DOF system with nonlinear coupling. The authors mainly used a combination of the analytical approach and numerical methods. The following analytical methods are commonly used: multiple scale method (Jo and Yabuno, 2009; Ji and Zhang, 2010; Cirillo *et al.*, 2017; Liu *et al.*, 2022), averaging method (Gendelman and Starosvetsky, 2007; Zhu *et al.*, 2004; Yang *et al.*, 2014; Febbo and Machado, 2013) harmonic balance method (Habib *et al.*, 2015; Peng *et al.*, 2012) and some combined techniques (Luongo and Zulli, 2012).

Many different aspects of the problem have been investigated and numerous kinds of DVA design were suggested. In the paper by Yu and Luo (2019), analytical solutions were obtained for periodic steady-state characteristics in a nonlinear vibration absorber under harmonic excitation. The stability and bifurcations analysis of periodic responses were obtained through eigenvalue analysis. Zhou *et al.* (2019) considered a DVA with negative stiffness, and their parameter optimization was conducted according to two tuning methodologies: the fixed points theory and the stability maximization criterion. Li and Zhang (2020) studied a 2-DOF system composed of a linear main structure under harmonic excitation and a TMD mass block connected by nonlinear with viscous damping. Special formula for a frequency of a tuned mass damper was suggested. A similar system was studied by Awrejcewicz *et al.* (2020) with the assumption that the frequency and amplitude of the excitation are not known. The noteworthy results based on experimental study were obtained in papers by Gatti *et al.* (2010), Kremer and Liu (2017) and Bronkhorst *et al.* (2018). In the paper by Islam and Jangid (2022), the performance of the LDVA was studied using multiple objective functions of the damped SDOF structure. Namely, the impact of the absorber damping ratio, frequency ratio, inertance ratio and structural damping ratio on its performance were investigated. Tuning of the DVA for such a system subjected to stationary white-noise earthquake excitation was discussed in the paper by Prakash and Jangid (2022).

At the same time, in the case of uncertain parameters, the problem is complex and numerical study does not give a complete picture of dynamics of the system. From the theoretical point of view, the problem cannot be considered closed because different aspects may be taken into consideration, and some of them are more (or less) important depending on the situation. In other words, there is no universal formula for DVA characteristics.

In the present paper, we discuss dynamics of a 2-DOF system which consists of a Duffing oscillator subjected to simple harmonic excitation with a linear DVA attached and uncertain parameters (frequency and amplitude of external excitation). The main attention is paid to the development of an analytical technique for determining absorber parameters that contribute to the maximum reduction in the amplitude of oscillations of the main system in the vicinity of resonant frequencies. To simplify the mathematical model, the Krylov-Bogolyubov averaging method is used. Then analysis of state-response maximal amplitudes was performed. Relations for determining the characteristics of the absorber are presented. Finally, some numerical examples illustrate the results obtained.

2. Description of the model and preliminary simplifications

The mechanical system under study consists of the Duffing oscillator with a hardening spring (primary system) with an attached linear DVA (Fig. 1).

The equations of motion of this system may be written in the following form

$$\begin{aligned} &m_1\ddot{x}_1 + c_a(\dot{x}_1 - \dot{x}_a) + k_{lin}x_1 + k_{nl}x_1^3 + k_a(x_1 - x_a) = F_0\cos\omega t \\ &m_a\ddot{x}_a + c_a(\dot{x}_a - \dot{x}_1) + k_a(x_a - x_1) = 0 \end{aligned} \tag{2.1}$$

where $x_1(t)$ and $x_a(t)$ are the displacements of the harmonically forced primary system and the absorber. The Duffing oscillator has a hardening spring ($k_{nl} > 0$), the damping coefficient and stiffness of the absorber are c_a and k_a, respectively. Introducing a new variable $x_2 = x_a - x_1$, system (2.1) may be rewritten in the following form

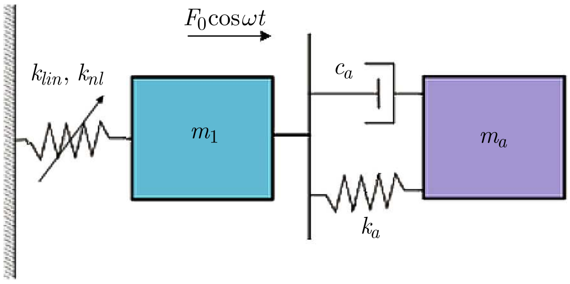

Fig. 1. Mechanical system

$$\begin{aligned}&(m_1+m_a)\ddot{x}_1+m_a\ddot{x}_2+k_{lin}x_1+k_{nl}x_1^3=F_0\cos\omega t\\&m_a\ddot{x}_1+m_a\ddot{x}_2+c_a\dot{x}_2+k_ax_2=0\end{aligned} \tag{2.2}$$

Let us introduce the dimensionless parameters and time by formulas

$$\begin{aligned}&\omega_1=\sqrt{\frac{k_{lin}}{m_1}} && \omega_a=\sqrt{\frac{k_a}{m_a}} && \mu=\sqrt{\frac{m_a}{m_1}} && q=\Big(\frac{\omega}{\omega_1}\Big)^2\\&\gamma=\frac{\omega_a^2}{\omega_1^2} && \widetilde{h}=\frac{c_a}{m_a\omega_1} && \widetilde{\alpha}=\frac{k_{nl}F_0^2}{k_{lin}} && \tau=\omega t\end{aligned} \tag{2.3}$$

Also, we introduce the dimensionless displacements by formulas

$$\widetilde{x}_1=\frac{x_1}{F_0} \qquad \widetilde{x}_2=\frac{\sqrt{\mu}}{F_0}x_2 \tag{2.4}$$

and now the equations of motion are in the following form

$$\mathbf{M}\widetilde{\mathbf{x}}''+\mathbf{D}\widetilde{\mathbf{x}}'+\mathbf{K}\widetilde{\mathbf{x}}=\mathbf{\Phi}(\tau,\widetilde{x}_2) \tag{2.5}$$

where the matrices $\mathbf{M}$, $\mathbf{D}$, $\mathbf{K}$, $\mathbf{\Phi}$ are defined according to formulas

$$\begin{aligned}&\mathbf{M}=q\begin{bmatrix}1+\mu & \sqrt{\mu}\\ \sqrt{\mu} & 1\end{bmatrix} && \mathbf{D}=\operatorname{diag}(0,h_\star) && \mathbf{K}=\operatorname{diag}(1,\gamma)\\&h_\star=\widetilde{h}\sqrt{q} && \mathbf{\Phi}=\begin{bmatrix}\cos\tau-\widetilde{\alpha}\widetilde{x}_1^3\\ 0\end{bmatrix} && \widetilde{\mathbf{x}}=\begin{bmatrix}\widetilde{x}_1\\ \widetilde{x}_2\end{bmatrix}\end{aligned} \tag{2.6}$$

Here, the prime denotes the derivative with respect to time τ. For convenience, the superscript "$\sim$" over x_1, x_2 is subsequently discarded. After the transformation

$$x_j=u_j\cos\tau+v_j\sin\tau \qquad x_j'=-u_j\sin\tau+v_j\cos\tau \qquad j=1,2 \tag{2.7}$$

equations (2.5) take the following form

$$\begin{aligned}&\cos\tau[\mathbf{M}\mathbf{v}'+(-\mathbf{M}+\mathbf{K})\mathbf{u}+\mathbf{D}\mathbf{v}]-\sin\tau[\mathbf{M}\mathbf{u}'+\mathbf{D}\mathbf{u}+(\mathbf{M}-\mathbf{K})\mathbf{v}]=\mathbf{\Phi}\\&\cos\tau(\mathbf{u}'+\mathbf{v})+\sin\tau(\mathbf{v}'-\mathbf{u})=-\sin\tau\mathbf{u}+\cos\tau\mathbf{v}\end{aligned} \tag{2.8}$$

Multiplying the previous system by the matrix

$$\begin{bmatrix}-\sin\tau & \cos\tau\\ \cos\tau & \sin\tau\end{bmatrix} \tag{2.9}$$

on the left throughout, we get the following equations

$$\begin{aligned}&\mathbf{M}\mathbf{u}'+\frac{1}{2}[(1-c_2)\mathbf{D}+s_2(\mathbf{M}-\mathbf{K})]\mathbf{u}+\frac{1}{2}[-s_2\mathbf{D}+(1-c_2)(\mathbf{M}-\mathbf{K})]\mathbf{v}=\mathbf{\Phi}_1\\&\mathbf{M}\mathbf{v}'+\frac{1}{2}[-s_2\mathbf{D}+(1+c_2)(\mathbf{K}-\mathbf{M})]\mathbf{u}+\frac{1}{2}[(1+c_2)\mathbf{D}+s_2(\mathbf{K}-\mathbf{M})]\mathbf{v}=\mathbf{\Phi}_2\end{aligned}\tag{2.10}$$

The expressions for the right-hand sides are described by formulas

$$\mathbf{\Phi}_1=\frac{1}{8}\begin{bmatrix}\phi_1\\0\end{bmatrix}\qquad\qquad \mathbf{\Phi}{=}\frac{1}{8}\begin{bmatrix}\phi_2\\0\end{bmatrix}\tag{2.11}$$

where

$$\begin{aligned}&\phi_1=-4s_2+\widetilde{\alpha}[(2s_2+s_4)u_1^3+3(1-c_4)u_1^2v_1+3(2s_2-s_4)u_1v_1^2+(3-4c_2+c_4)v_1^3]\\&\phi_2=4(1+c_2)-\widetilde{\alpha}[(3+4c_2+c_4)u_1^3+3(2s_2+s_4)u_1^2v_1+3(1-c_4)u_1v_1^2+(2s_2-s_4)v_1^3]\\&s_2=\sin 2\tau\qquad c_2=\cos 2\tau\qquad s_4=\sin 4\tau\qquad c_4=\cos 4\tau\end{aligned}$$

Now, we apply the Krylov-Bogolyubov averaging method. Assuming that u, v vary slowly with time τ, we obtain the averaged equations

$$\begin{aligned}&\mathbf{M}\mathbf{u}'+\frac{1}{2}[\mathbf{D}\mathbf{u}+(\mathbf{M}-\mathbf{K})\mathbf{v}]=\frac{3}{8}\begin{bmatrix}\widetilde{\alpha}v_1(u_1^2+v_1^2)\\0\end{bmatrix}\\&\mathbf{M}\mathbf{v}'+\frac{1}{2}[(\mathbf{K}-\mathbf{M})\mathbf{u}+\mathbf{D}\mathbf{v}]=\frac{1}{8}\begin{bmatrix}4-3\widetilde{\alpha}u_1(u_1^2+v_1^2)\\0\end{bmatrix}\end{aligned}\tag{2.12}$$

Let us find the stationary points of system (2.12). Technically, it is easier to do this by introducing the complex variables $\mathbf{z}=\mathbf{u}+\mathrm{i}\mathbf{v}$. Then equations (2.12) take the following form

$$\mathbf{M}\mathbf{z}'+\frac{1}{2}(\mathbf{B}+\mathrm{i}\mathbf{D})z=\frac{1}{2}\mathrm{i}\begin{bmatrix}1-\alpha z_1^2\overline{z}_1\\0\end{bmatrix}\qquad \mathbf{M}\overline{z}'+\frac{1}{2}(\mathbf{B}+\mathrm{i}\mathbf{D})\overline{z}=\frac{1}{2}\mathrm{i}\begin{bmatrix}-1+\alpha z_1\overline{z}_1^2\\0\end{bmatrix}\tag{2.13}$$

Here, the following notions are introduced

$$\mathbf{K}-\mathbf{M}\triangleq\mathbf{B}=\begin{bmatrix}b_{11}&b_{12}\\b_{12}&b_{22}\end{bmatrix}\tag{2.14}$$

where

$$b_{11}=1-q(1+\mu)\qquad b_{12}=-\mu q\qquad b_{22}=\gamma-q\qquad \alpha=\frac{3}{4}\widetilde{\alpha}$$

The condition $z'=0$ leads to the following system of algebraic equations

$$\begin{aligned}&b_{11}z_1+b_{12}z_2-1+\alpha z_1^2\overline{z}_1=0\qquad b_{12}z_1+(b_{22}-\mathrm{i}h)z_2=0\\&CC_1=0\qquad CC_2=0\end{aligned}\tag{2.15}$$

where CC_1, CC_2 are corresponding complex conjugates.

Expressing the variable z_2 from the second equation, we come to a pair of nonlinear equations with respect to z_1, $\overline{z}_1$

$$(b_{22}-\mathrm{i}\widetilde{h})z_1^2\overline{z}_1+(b_{11}b_{22}-b_{12}^2-\mathrm{i}\widetilde{h}b_{11})\overline{z}_1-b_{22}+\mathrm{i}\widetilde{h}=0\qquad CC=0\tag{2.16}$$

Further, we rewrite equalities (2.16) in the form

$$\begin{aligned}&[\alpha r(b_{22}-\mathrm{i}\widetilde{h})+\det\mathbf{B}-\mathrm{i}\widetilde{h}b_{11}]z_1=b_{22}-\mathrm{i}\widetilde{h}\\&[\alpha r(b_{22}+\mathrm{i}\widetilde{h})+\det\mathbf{B}+\mathrm{i}\widetilde{h}b_{11}]\overline{z}_1=b_{22}+\mathrm{i}\widetilde{h}\qquad r=z_1\overline{z}_1\end{aligned}\tag{2.17}$$

Now, multiplying separately the left- and right-hand sides of the two last equalities, we obtain the real-valued equation

$$\begin{aligned} &rq^4 - r[2\alpha r + 2(1+\mu)\gamma - h(1+\mu)^2 + 2]q^3 + \{\alpha^2 r^3 + 2\alpha[(2+\mu)\gamma - (1+\mu)h + 1]r^2 \\ &\quad + [(1+\mu)^2\gamma^2 + 2(2+\mu)\gamma - 2(1+\mu)h + 1]r - 1\}q^2 \\ &\quad - \{\alpha^2(2\gamma - h)r^3 + 2\alpha[(1+\mu)\gamma^2 + 2g - h]r^2 + [2(1+\mu)\gamma^2 + 2\gamma - h]r - 2\gamma + h\}q \\ &\quad + \gamma^2[r(\alpha r + 1)^2 - 1] = 0 \qquad h = \widetilde{h}^2 \end{aligned} \tag{2.18}$$

This equation determines the frequency-amplitude hyper-surface in six-dimensional space (related to equation (2.18)), where the square of amplitude of the primary mass $r = (u_1^2 + v_1^2)$ depends also on dimensionless parameters μ, α, h, γ. When the mechanical parameters of both masses are known, we have a traditional frequency-amplitude curve. Note that in the case $\alpha = 0$ (linear system), equation (2.18) is equivalent to the well-known classical form (Den Hartog, 1934).

Remark 1. We do not present here the conditions for stability of stationary points, as well as the conditions for the existence of three positive roots for r of polynomial (2.18). These conditions can be obtained in a similar manner, as it was done in the paper by Awrejcewicz *et al.* (2020).

3. Mitigation of the responses of the main mass

Assuming that the parameters m_1, k_{lin}, k_{nl} of the main system are given, as well as the mass of the absorber, we want to choose the damping coefficient and absorber stiffness in order to reduce the maximum level of possible oscillation amplitude in the vicinity of resonant frequencies (i.e., the peaks of the frequency-amplitude curve). With respect to dimensionless parameters, which means that μ and α are known, while the parameters h and γ should be determined.

The optimal case holds place when the ordinates of the peaks coincide, which geometrically means that at these points the tangents $r = r_1$, $r = r_2$ to the curve $r(q)$ coincide. In other words, the equation $f(r_0, q) = 0$ ($r_0 =$ const) has three or four different real roots if $r_0 < r_{max}$, and two multiple roots if $r_0 = r_{max}$ (Fig. 2). Here r_{max} corresponds to the maximum value of r.

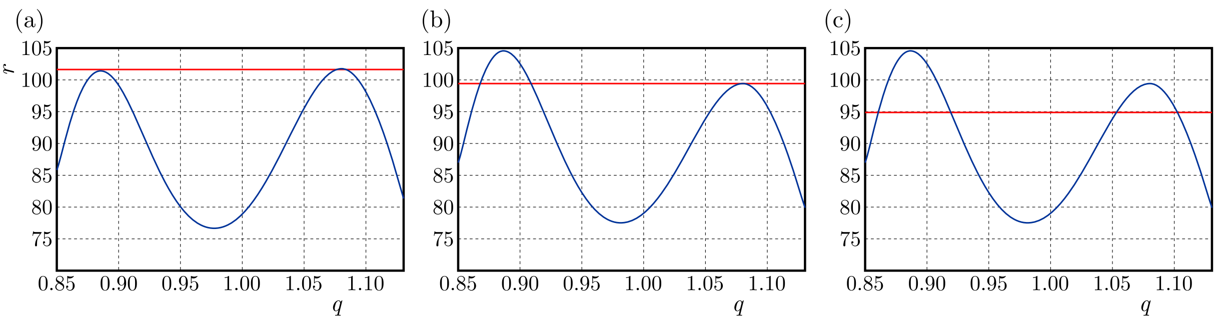

Fig. 2. Different number of solutions of equation $f(r_0, q) = 0$ depending on values of h and γ ($\mu = 0.02$, $\alpha = 0.002$): (a) $r_0 = r_{max}$, (b), (c) $r_0 < r_{max}$

The conditions for a fourth-degree polynomial

$$P(\xi) = a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0 \qquad a_1 a_3 \neq 0 \tag{3.1}$$

to have two roots of multiplicity are

$$\psi_1 \triangleq a_0 a_3^2 - a_1^2 a_4 = 0 \qquad \psi_2 \triangleq a_3^3 - 4a_2 a_3 a_4 + 8a_1 a_4^2 = 0 \tag{3.2}$$

Substituting the corresponding expressions for the coefficients a_j, we have

$$\psi_1(r) = \sum_{j=0}^{3} \psi_{1j} r^j \qquad \psi_2(r) = \sum_{j=0}^{6} \psi_{2j} r^j \tag{3.3}$$

where

$$\begin{aligned}
&\psi_{10} = 4[2(1-\mu)\gamma + h(1-2\mu-\mu^2)+2] \qquad \psi_{13} = 8\mu^2\alpha^2 h \\
&\psi_{11} = 4(1+\mu)^2\gamma^4 - 4h(1+\mu)^3\gamma^3 - [4+4\mu(1+\mu)h - (1+\mu)^4h^2]\gamma^2 + 8h\gamma - 2h^2 \\
&\psi_{12} = 4h(1+\mu)^3\gamma^3 - [(1+\mu)^4h^2 - 4\mu 1+\mu)(1-\alpha)h + 8\alpha]\gamma^2 + 4h(1-4\alpha)\gamma - h^2(1-4\alpha) \\
&\psi_{20} = 4[-2(1-\mu)\gamma + h(1-2\mu-\mu^2)+2 \\
&\psi_{21} = -4(1+\mu)^2\gamma^4 + 4h(1+\mu)^3\gamma^3 + [4+4\mu(1+\mu)h - (1+\mu)^4h^2]\gamma^2 - 8h\gamma + 2h^2 \\
&\psi_{22} = -4h(1+\mu)^3\gamma^3 + [(1+\mu)^4h^2 - 4\mu(1+\mu)(1-\alpha)h + 8\alpha]\gamma^2 \\
&\qquad + 4h(1-4\alpha)\gamma - h^2(1-4\alpha) \\
&\psi_{23} = -2\alpha\{4h(1+\mu)^3\gamma^3 - [(1+\mu)^4h^2 - 6\mu(1+\mu)h + 2\alpha]\gamma^2 - 4h(2-\alpha)\gamma + h^2(2-\alpha)\} \\
&\psi_{24} = -\alpha^2 h\{4(1+\mu)^3\gamma^3 + (1+\mu)[12\mu - (1+\mu)^3h]\gamma^2 - 24\gamma + 6h\} \\
&\psi_{25} = -4\alpha^3 h[\mu(1+\mu)\gamma^2 - 4\gamma + h] \qquad \psi_{26} = \alpha^4 h(4\gamma - h)r^6
\end{aligned} \tag{3.4}$$

Thus, we have two conditions $\psi_1 = 0$, $\psi_2 = 0$ connecting the function r and the optimization parameters h, γ. Considering $\psi_1 = 0$ as an implicit function of $r(h,\gamma)$ and $\psi_2 = 0$ as a constraint (or vice versa), we arrive at the problem of finding a conditional extremum. Compiling a linear combination $\psi(y,h,\gamma) = \psi_1 + \lambda\psi_2$ (λ is the Lagrange multiplier), we obtain a necessary condition for the existence of an extremum in the following form

$$\Delta(y,h,\gamma,\mu,\alpha) = \frac{\partial\psi_1}{\partial h}\frac{\partial\psi_2}{\partial\gamma} - \frac{\partial\psi_1}{\partial\gamma}\frac{\partial\psi_2}{\partial h} = 0 \tag{3.5}$$

Thus, for given values of the parameters μ and α, we have a system of three algebraic equations from which the corresponding values of the DVA parameters can be found numerically. Since the polynomials ψ_1, ψ_2, Δ have a very high degree, the direct solution of such a system of equations is computationally costly and may need an extra effort to avoid errors. Therefore, it makes sense to localize suitable ranges of values of γ and h firstly. It is convenient to do this geometrically, that is, to approximately determine the intersection point of the surfaces

$$\psi_1(\gamma,h,r) = 0 \qquad \psi_2(\gamma,h,r) = 0 \qquad \Delta(\gamma,h,r) = 0 \tag{3.6}$$

as shown in Fig. 3.

By choosing the values h_0, γ_0, r_0 found from Fig. 3 as an initial approximation, it is much easier to determine appropriate corrections. For example, substituting $h_1 = h_0 + \delta_h$, $\gamma_1 = \gamma_0 + \delta_\gamma$, $r_1 = r_0 + \delta_r$ into expressions ψ_1, ψ_2, Δ, and expanding the latter into the Taylor series in δ_h, δ_γ, δr, we can limit ourselves to a linear approximation while finding the refined approximation (with subsequent comparison of errors for thr left-hand sides of system (3.5) in both cases). In particular, for $\mu = 0.05$, $\alpha = 0.002$ we find $h \approx 0.07$, $\gamma \approx 0.973$, the corresponding FAC is shown in Fig. 4. The evolution of FAC with optimized DVA parameters depending on the value of α is presented in Fig. 5. With a growth of α, the value of γ slowly increases and the peaks go down.

Remark 2. Due to the fact that polynomials in system (3.5) are of extremely high degrees, it seems impossible to obtain explicit expressions for h, γ, r as functions of μ, α. These expressions can be approximately obtained in the form of asymptotic expansions. However,

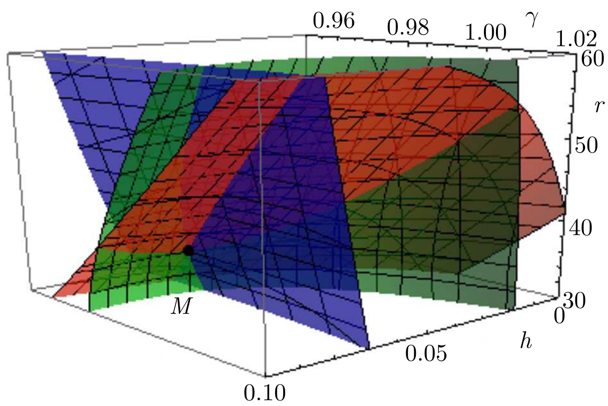

Fig. 3. Geometrical interpretation of system (3.5), where orange, green and blue surfaces correspond to $\psi_1 = 0$, $\psi_2 = 0$ and $\varDelta = 0$, $\mu = 0.05$, $\alpha = 0.002$

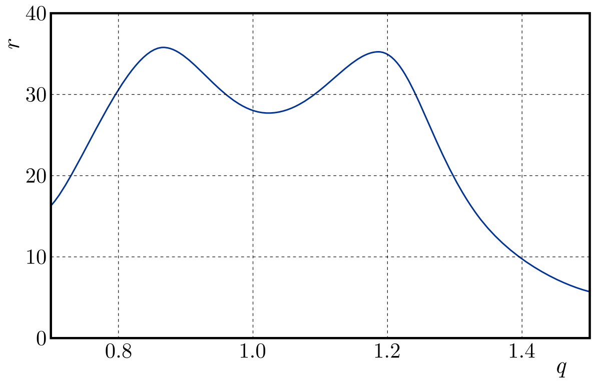

Fig. 4. The frequency-amplitude curve with optimized DVA parameters (based on Eq. (2.18), $\mu = 0.05$, $\alpha = 0.002$, $h = 0.07$, $\gamma = 0.973$

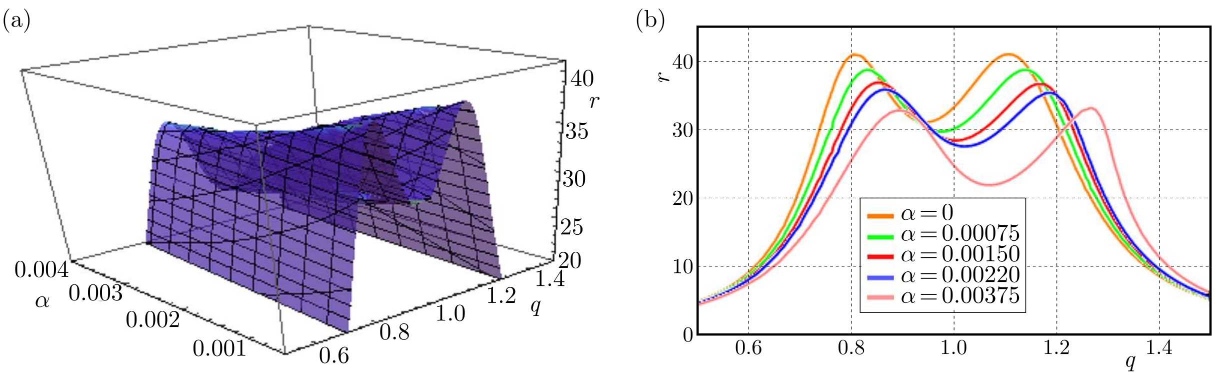

Fig. 5. Dependence of the responses of primary mass on the parameter α for $\mu = 0.05$

taking into account that the problem contains three small parameters (μ, F_0, k_{nl}), obtaining such expansions essentially depends on how these parameters relate to each other (for example, k_{nl} can be considered an independent small parameter, it can be considered proportional to μ or μ^2 etc.). The conscientious obtaining of such formulas is a subject of separate consideration.

4. Discussion and numerical validation

The procedure described above makes it possible to make an optimal choice of the absorber parameters, assuming that the parameter α is known. As can be seen from formulas (2.3), this parameter depends not only on the nonlinear stiffness k_{nl} of the main system, but also on the amplitude F_0 of the external force. If its exact value is unknown, then what corrections should be made? This issue was discussed, in particular, in (Habib *et al.*, 2015), where the authors proposed the use of a nonlinear absorber with "mirror" characteristics (proportional to characteristics of the main system). Such an approach seems to be logical, but the complexity of analysis significantly increases. For example, a cubic polynomial on r in formula (2.18) transforms into a polynomial of the ninth degree, and even the problem of determining the number of real roots is mathematically very difficult (and the numerical approach is not very reliable due to the large number of unknown parameters). Thus, the question arises: is it possible to use a linear absorber in conditions where the exact value of F_0 is vague, but the interval of possible values of the amplitude is known, i.e. $F_0 \in [F_1, F_2]$. If this interval is not too wide, say $F_2/F_1 \leqslant 2$, then taking into account that with a growth of α, the value of γ also increases (and to a lesser extent the value of h), as can be seen in Fig. 5, then we suggest that when choosing the value of γ we should take the maximum possible value of the amplitude of the external excitation F_2. This is illustrated in Figs. 6 and 7.

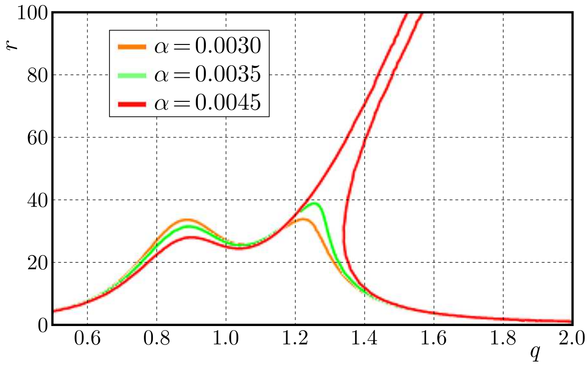

Fig. 6. Evolution of the responses as F_0 goes up for $h = 0.07$, $\gamma = 0.999$

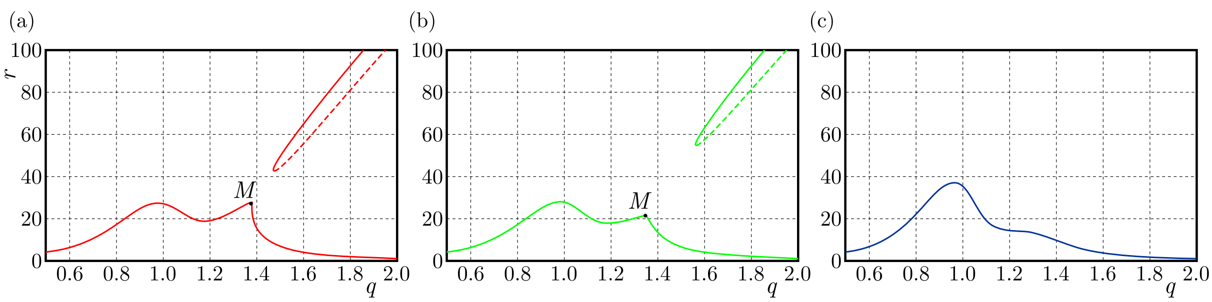

Fig. 7. Shape of the FAC when DVA parameters are chosen for α corresponding to $F_0 = F_2$ for $h = 0.071$: (a) $\alpha = 0.008$, $\gamma = 1.109$, (b) $\alpha = 0.008$, $\gamma = 1.12$, (c) $\alpha = 0.005$, $\gamma = 1.12$

Suppose that $\alpha \in [0.002, 0.008]$ and the "expected" value of α is equal to 0.003. Then calculated values for the absorber are: $\gamma \approx 0.999$, $h \approx 0.07$. The corresponding FAC is shown in Fig. 6. It has one branch and two equal peaks at $r \approx 34$. The allowable frequency range is wide enough. If the amplitude grows by 10%, the maximal amplitude increases up to 40 units, but the curve remains "robust" – no bifurcations, no blow-up, etc. However with a further growth by 10%, we get bifurcations and a significant increase in the amplitude of oscillations for primary mass. From the other side, calculating the absorber parameters for the upper limit $\alpha = 0.008$, we get $\gamma \approx 1.09$, $h \approx 0.071$. The FAC for such values is presented in Fig. 7. It looks the same as in the case of the linear system, the value of r_{max} decreases down to ≈ 28, and there is some distance between the right peak and the upper branch, thus the "jump" from the lower branch to the upper one as a result of some occasional perturbations is not expected. Moreover, if we choose $\gamma = 1.12$, then r_{max} grows up to 30 units, but the point M now is far enough from the upper branch (Fig. 7).

In order to verify the obtained analytical results, we carried out numerical integration of averaged system (2.12) and equations (2.5). Also, in order to compare the efficiency of the absorber based on the proposed approach with other results, we took of the parameters $\mu = 0.05$, $\alpha = (3/4) \cdot 0.013$, as it was done in the article (Habib *et al.*, 2016) in Fig. 4a. Those authors considered a nonlinear absorber, however the values for h and γ were taken as optimal values for the linear system, namely $\gamma = 0.9524^2$, $h = (2 \cdot 0.134)^2\gamma$. The corresponding cubic component of stiffness of the absorber was taken $\beta = (4/3) \cdot 0.0851 \cdot \alpha$[1]. The maximum dimensionless response of main mass was between 5.5 and 6.0. At the same time, solving system (3.6) (or using its geometrical interpretation like it is shown in Fig. 3) for the LDVA we found: $h = 0.081$, $\gamma = 1.15$, $q = 1.39$. Substituting these values into system (2.15), we have $u_1 = -2.624$, $u_2 = -0.50619$, $v_1 = 2.70287$, $v_2 = -2.79266$ (two other solutions are complex with respect to $\mathbf{u}$, $\mathbf{v}$). After this, we integrate the averaged equations for both cases LDVA and NLDVA (with identical initial values), the results are shown in Fig. 8a. The distance from the origin to the attraction point is shorter with our choice of parameters of the absorber. Also, as one can see, trajectories with different initial values tend to the attraction point, and coordinates of this point correlate with the found values of u_1, v_1. The fact that the averaged equations describe well the behavior of the solutions of the original system is confirmed by integrating equations (2.5).

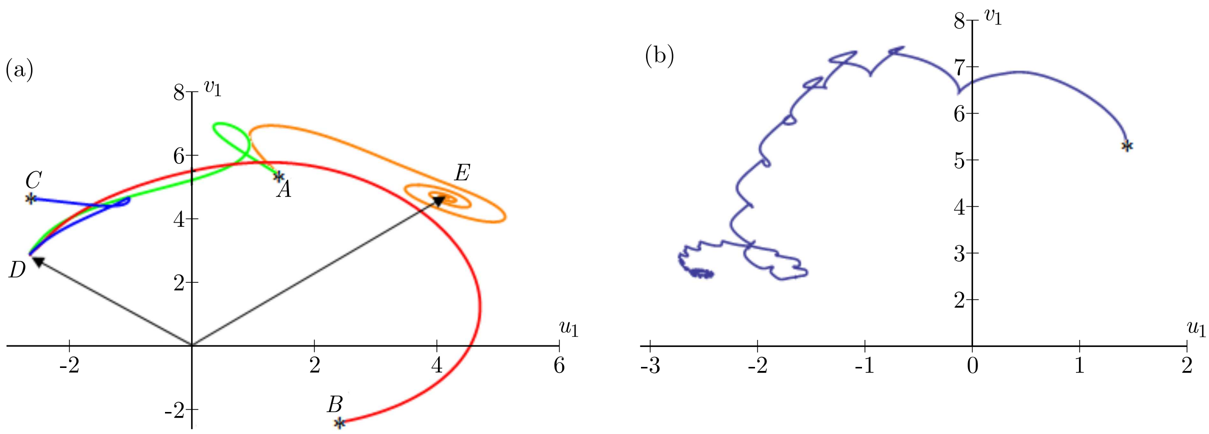

Fig. 8. (a) Phase trajectories for system (2.12) with different initial values (points A, B, C) and the stationary point D, $\mu = 0.05$, $\alpha = 0.00975$. The point E is the attraction point for NLDVA with cubic nonlinearity and parameters of the absorber taken according to (Habib *et al.*, 2016); (b) trajectory for non-averaged system (2.10) with the initial point A

As can be seen in Fig. 9a, the stationary point of system (2.12) corresponds to the limit cycle of system (2.5). The projection of the phase trajectory on the plane x_1, x_1' which corresponds to the NLDVA with values according to (Habib *et al.* 2016), see Fig. 4a, is shown in Fig. 9b, and a comparison of two time histories for the dimensionless response of main mass is presented in Fig. 9c.

[1]The averaged equations for such a case will differ from (2.5) by appearing the term $-\beta x_2^3$ in the second row of the matrix $\mathbf{\Phi}$.

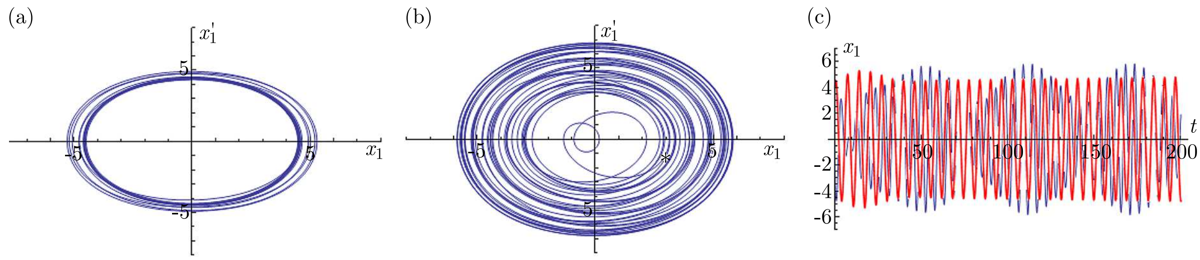

Fig. 9. (a) Projection of the phase trajectory on the plane x_1x_1' for dimensionless system (2.5) with the proposed LDVA parameters, the frequency ratio corresponds to the maximum response value $q = 1.39$; (b) trajectory corresponding to the NLDVA, $q = 1.25$; (c) comparison of two time histories for the LDVA and NLDVA

5. Conclusions

The article considers the problem of determining the parameters of a DVA connected to a Duffing's oscillator, which is under the influence of a periodic external excitation. The goal is to reduce the maximum possible oscillation amplitude of the main system under conditions of uncertainty (frequency ratio and external excitation amplitude) in the vicinity of resonant frequencies. We paid special attention to the development of an analytical procedure for selection the DVA parameters. It is shown that an appropriate choice of absorber stiffness depends significantly on the amplitude of the external action. In the case when this amplitude can take values from a certain range, it is advisable to focus on the upper limit of this range when choosing the absorber frequency.

The future work will be related to obtaining asymptotic formulas for the absorber parameters, more detailed comparison of the efficiency of using linear and non-linear absorbers and estimating the region of attraction.

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