JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
61, 2, pp. 407-414, Warsaw 2023
https://doi.org/10.15632/jtam-pl/163169

ANALYSIS OF OUT-OF-PLANE FREE VIBRATION OF SINGLE DAMAGED CURVED BEAM BASED ON PRECISE ALGORITHM OF STRUCTURAL MECHANICS

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Based on a dynamic discrete model of an out-of-plane curved beam with a constant curvature, eigen-properties of the spatial curved beam structure in undamaged and damaged configurations are considered in this paper. In the literature, based on the equivalent section reduction method, a distributed damage modeling method is proposed. Accoding to Euler-Bernoulli beam theory, the stiffness matrix of shear, bending and torsion coupling is derived. Combined with the lumped mass matrix and the characteristic equation of the multi degree of freedom system, natural frequencies of the undamaged and damaged structures are calculated.

Keywords: out-of-plane curved beam, damaged, stiffness matrix, free vibration

1. Introduction

A curved beam structure is very important and widely used in many engineering fields. In practical engineering applications, curved beams often suffer from crack damage, which leads to structural damage. Based on Euler-Bernoulli hypothesis, if the section of a curved beam remains unchanged and is double symmetric, in-plane vibration and out-of-plane vibration are independent of each other (Zhao *et al.*, 2006). For some specific structures, such as urban ring overpass, the out-of-plane vibration is the main mode of motion, and cracks and other damage characteristics will occur in this direction, which will affect vibration characteristics of spatial curved beams (Hu *et al.*, 2007; Mao *et al.*, 2020). Therefore, vibration analysis of space curved beams with cracks is very significant for the research of crack identification, and it is more important to find accurate theoretical solutions.

Modeling a curved beam with damage is a major problem. Damage modeling of curved beams can be found in some papers (Friswell and Penny, 2002; Dessi and Camerlengo, 2015; Petroski, 1981; Pau *et al.*, 2011). Based on the above methods, researchers have published many studies on cracked beam structures with a large number of results, see for example (Wang *et al.*, 2018; Anifantis and Dimarogonas, 1983; Dimarogonas, 1996). The research on free vibration of curved beams can be divided into in-plane vibration and out-of-plane vibration studies, most of which focus on in-plane vibration, while out-of-plane vibration is relatively rare. Howson and Jemah (1999) proposed an effective method to solve the out-of-plane natural frequency of a plane structure composed of bending Timoshenko beams. Ishaquddin *et al.* (2016) studied the out-of-plane vibration of Timoshenko and Euler-Bernoulli curved beam elements. The performance of coupled and independent curved beam elements was evaluated. Caliò *et al.* (2014, 2016) carried out a primitive and extensive parametric analysis of the in-plane and out-of-plane dynamic behavior of a single arch. They studied dynamic characteristics of a space structure composed of circular arches under undamaged and damaged conditions. The influence of damage parameters on the natural frequency of vibration was shown. Zare (2019) used on the differential quadrature element method (DQEM), to determine intrinsic frequencies of the cracked beams.

The finite element method can effectively improve computational efficiency and obtain an approximate solution close to the exact solution. However, by its principle, its total stiffness matrix is usually set into a bar matrix distributed along the diagonal, which contains a large number of zero-coupling terms, which will inevitably bring some errors. The method in this paper is a semi-analytical method that takes into account the coupling effects between non-adjacent nodes and can effectively analyze vibration modes of the damaged structure through parameter control.

This paper describes a dynamical model of a spatial curved beam structure, and investigates self-oscillation characteristics of a spatial lossless curved beam structure and a damaged structure by establishing the distributed damage model of a out-of-plane curved beam. It mainly includes two parts: 1) based on the internal force subsection function of the space statically indeterminate structure, the accurate global stiffness matrix and the concentrated mass matrix considering the rotation effect are obtained through integral calculation; 2) based on the characteristic equation of the equation of motion of the multi degree of freedom system, a MATLAB program is compiled to obtain the corresponding eigenvalues and eigenvectors of each order.

2. Theoretical analysis

Figure 1 shows a out-of-plane curved beam model. Because of the special structural form of the space curved beam, the column coordinate system should be used when analyzing the problem. The corresponding out-of-plane generalized load forms include the vertical force F_z, radial couple F_ρ and circumferential torque F_θ. These three forces correspond to three generalized displacements, namely the vertical displacement u_z, angular displacement u_ρ of the section around the radial axis, and the torsional angular displacement u_θ of the section around the circumferential axis. These three generalized displacements can be used as dynamic degrees of freedom for out-of-plane modal analysis of curved beams. Any section of the whole beam only produces three kinds of internal forces: the shear force Q_z perpendicular to the plane of the curved beam, the bending moment M_ρ around the radial axis, and the torque T_θ around the circumferential axis. The curvature radius of the curved beam is R, the section shear stiffness coefficient is k, the section area is A, the inertia moment of z axis is I_z, the polar inertia moment is I_p, the section height is h, and the section width is b. The elastic modulus of the material is E, the shear modulus is G, Poisson's ratio is ν, and the density is ρ.

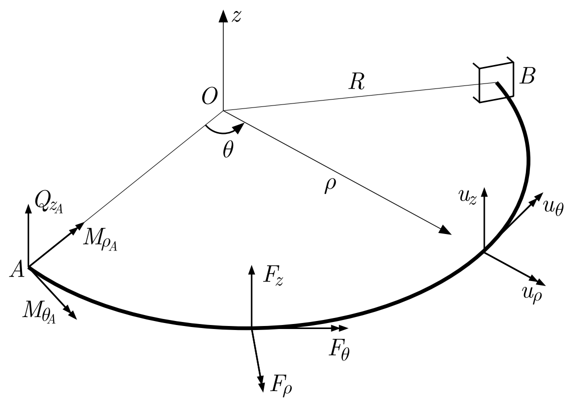

Fig. 1. Statical analysis model of a curved beam

In this paper, a distributed damage model of curved beams is studied. In order to show the weakening effect of damage on section characteristics of curved beams, the curved beams are divided into three sections, that is, $[0,\theta_1]$, $[\theta_1,\theta_2]$ and $[\theta_2,\theta_3]$ are denoted as the cross-section area A_1, elastic modulus E_1, shear modulus G_1 in the first and third sections, and A_2, E_2, G_2 in the second section. According to the modeling characteristics of the distributed damage model, and the damage section is defined as

$$\alpha = \frac{h_d}{h} \qquad \beta = \frac{\theta_2 - \theta_1}{2\varphi} \tag{2.1}$$

where α is the damage degree coefficient, h_d is the section height of the damaged beam element, and β characterizes the section position of damage.

3. Free vibration analysis of an out-of-plane curved beam

In the undamped modal analysis of curved beam structures, the required structural characteristic matrices are stiffness matrix $\mathbf{K}$ and mass matrix $\mathbf{M}$. Calculating the mode of a curved beam belongs to solving an eigenvalue problem from the mathematical point of view. The control equation of its structural characteristics is

$$\mathbf{K}\boldsymbol{\phi} = \lambda\mathbf{M}\boldsymbol{\phi} \tag{3.1}$$

In the formula, λ is the eigenvalue corresponding to the natural vibration frequency of the structure, and $\boldsymbol{\phi}$ is the structural modal vector corresponding to the eigenvector.

3.1. Stiffness matrix integration of the damaged curved beam

According to structural mechanics, the stiffness and flexibility matrix are reciprocal matrices, and the specific relationship can be expressed as follows

$$\mathbf{K} = \mathbf{D}^{-1} \tag{3.2}$$

It can be seen from Eq. (3.2) that the stiffness matrix can be obtained by the inverse solution of the flexibility matrix. According to the node numbering principle, the number sequence of the node degrees of freedom of the curved beam structure is: 1 – vertical displacement, 2 – section rotation displacement around the radial axis, 3 – section rotation displacement around the circumferential axis. The relationship between the node force and node displacement of the whole curved beam system can be expressed as

$$\mathbf{F} = \mathbf{K}\mathbf{u} \tag{3.3}$$

Formula (3.3) is multiplied by $\mathbf{K}^{-1}$ on both sides to get

$$\mathbf{u} = \mathbf{D}\mathbf{F} \tag{3.4}$$

The solution of the flexibility matrix can be decomposed into nine forms of matrix superposition. The nine kinds of matrices are the vertical uncoupled flexibility matrix $\mathbf{D}_{zz}$, rotational uncoupled flexibility matrix $\mathbf{D}_{\rho\rho}$, torsional uncoupled flexibility matrix $\mathbf{D}_{\theta\theta}$, vertical rotational coupling flexibility matrix $\mathbf{D}_{\rho z}$, vertical torsional coupling flexibility matrix $\mathbf{D}_{\theta z}$, rotational vertical coupling flexibility matrix $\mathbf{D}_{z\rho}$, rotational torsional coupling flexibility matrix $\mathbf{D}_{\theta\rho}$, torsional vertical coupling flexibility matrix $\mathbf{D}_{z\theta}$, torsional rotational coupling flexibility matrix $\mathbf{D}_{\rho\theta}$

$$\mathbf{D} = \mathbf{D}_{zz} + \mathbf{D}_{\rho\rho} + \mathbf{D}_{\theta\theta} + \mathbf{D}_{\rho z} + \mathbf{D}_{\theta z} + \mathbf{D}_{z\rho} + \mathbf{D}_{\theta\rho} + \mathbf{D}_{z\theta} + \mathbf{D}_{\rho\theta} \tag{3.5}$$

The method in this paper is applicable to find the out-of-plane vibration solutions for all boundary conditions of curved beams, and this article takes two-end clamped out-of-plane curved beams as the object of study. The solution of the flexibility matrix of the overall structure can be converted into a solution of the nine submatrices respectively. Since the curved beam is divided into three segments, the following are discussed respectively:

The radial unit load $P = 1$ acts on $[0, \theta_1]$. The internal force function of a statically indeterminate structure is established based on the geometric relationship of the structure:

— for $0 \leqslant \delta \leqslant \theta_p$

$$\begin{aligned} QL(\delta) &= Q_{Z_A} \\ ML(\delta) &= Q_{Z_A} R \sin\delta + M_{\rho_A} \cos\delta - T_{\theta_A} \sin\delta \\ TL(\delta) &= Q_{Z_A} R(1 - \cos\delta) + M_{\rho_A} \sin\delta - T_{\theta_A} \sin\delta \end{aligned} \tag{3.6}$$

— for $\theta_p \leqslant \delta \leqslant \varphi$

$$\begin{aligned} QR(\delta) &= Q_{Z_A} + 1 \\ MR(\delta) &= Q_{Z_A} R \sin\delta + M_{\rho_A} \cos\delta - T_{\theta_A} \sin\delta + R\sin(\theta - \theta_p) \\ TR(\delta) &= Q_{Z_A} R(1 - \cos\delta) + M_{\rho_A} \sin\delta - T_{\theta_A} \sin\delta + R[1 - \cos(\theta - \theta_p)] \end{aligned} \tag{3.7}$$

The three internal force subsection functions of a statically determinate curved beam structure under the action of an out-of-plane unit load are:

— under the action of the unit vertical force for $\delta \leqslant \theta_{\overline{p}}$

$$\overline{QL}(\delta) = 0 \qquad \overline{ML}(\delta) = 0 \qquad \overline{TL}(\delta) = 0 \tag{3.8}$$

and for $\delta \geqslant \theta_{\overline{p}}$

$$\overline{QR}(\delta) = 1 \qquad \overline{MR}(\delta) = R\sin(\delta - \theta_{\overline{p}}) \qquad \overline{TR}(\delta) = R[1 - \cos(\delta - \theta_{\overline{p}})] \tag{3.9}$$

— under the action of the unit couple for $\delta \leqslant \theta_{\overline{p}}$

$$\overline{QL}(\delta) = 0 \qquad \overline{ML}(\delta) = 0 \qquad \overline{TL}(\delta) = 0 \tag{3.10}$$

and for $\delta \geqslant \theta_{\overline{p}}$

$$\overline{QR}(\delta) = 0 \qquad \overline{MR}(\delta) = -\cos(\delta - \theta_{\overline{p}}) \qquad \overline{TR}(\delta) = -\sin(\delta - \theta_{\overline{p}}) \tag{3.11}$$

— under the action of the unit torque for $\delta \leqslant \theta_{\overline{p}}$

$$\overline{QL}(\delta) = 0 \qquad \overline{ML}(\delta) = 0 \qquad \overline{TL}(\delta) = 0 \tag{3.12}$$

and for $\delta \geqslant \theta_{\overline{p}}$

$$\overline{QR}(\delta) = 0 \qquad \overline{MR}(\delta) = -\sin(\delta - \theta_{\overline{p}}) \qquad \overline{TR}(\delta) = \cos(\delta - \theta_{\overline{p}}) \tag{3.13}$$

In Eqs. (3.6)-(3.13), θ_p is the angle of the super-static curved beam structure subject to the unit tip load, $\theta_{\overline{p}}$ is the angle beam structure subject to the tip imaginary unit load. QL, ML, TL and QR, MR, TR are the three internal force functions on the left and right sides of the action force, respectively.

The above functions are piecewise integrated and summed using the displacement integral formula

$$d_{ij} = \sum k \int \frac{\overline{Q} Q_P}{GA}\, ds + \sum \int \frac{\overline{M} M_P}{GI}\, ds + \sum \int \frac{\overline{T} T_P}{GI_\rho}\, ds \tag{3.14}$$

The obtained formula is integrated into the flexibility matrix of an order according to the coordinate numbering. According to the numbering, the obtained results are integrated into the $3n \times 3n$ order flexibility matrix. Similarly, formulas (3.6)-(3.14) are used to calculate the displacement summation in the cases $\theta_p \in [\theta_1, \theta_2]$ and $\theta_p \in [\theta_2, \varphi]$, respectively. The interval $[\theta_1, \theta_2]$ is represented as the damaged element, and the two ends of the element are given in polar angular coordinates. The section parameters of the element are changed to characterize the section distributed damage of the curved beam structure.

According to equation (3.2), the exact stiffness matrix with damage can be obtained by inversion of the matrix $\mathbf{D}$. The numerical distribution of the stiffness matrix $\mathbf{K}$ is shown in Fig. 2.

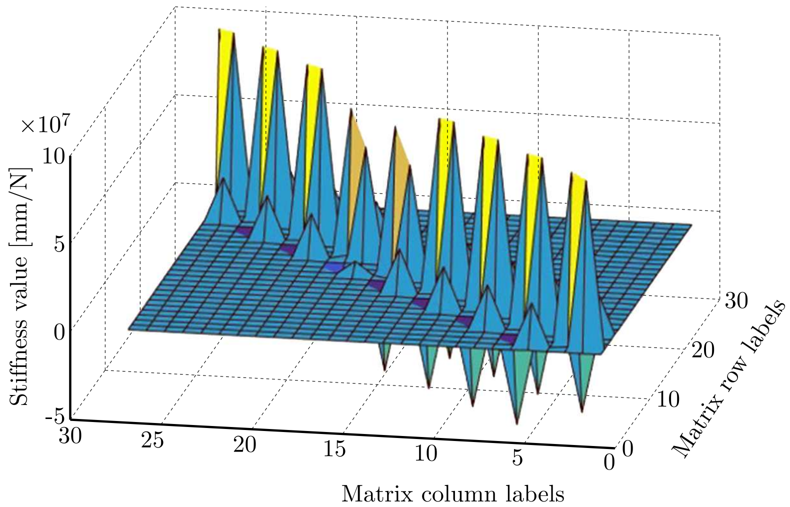

Fig. 2. Numerical distribution of the stiffness matrix with damage

It can be seen from Fig. 2 that the highest peak on the main diagonal is the coefficient related to rotational stiffness, which is much larger than other translational stiffness values. The abrupt change of the peak value is the damage position of the curved beam structure, which is due to the change of stiffness by reduction of the element section.

3.2. Mass matrix integration of out-of-plane curved beams

The dynamic model of the curved beam structure can be simplified as the mass discrete structural system shown in Fig. 3. The beam body is a non-open thin-walled section, without considering the influence of section warping. To illustrate the problem, the section form is assumed to be a rectangular section, while the corresponding stiffness characteristics of the whole beam are retained.

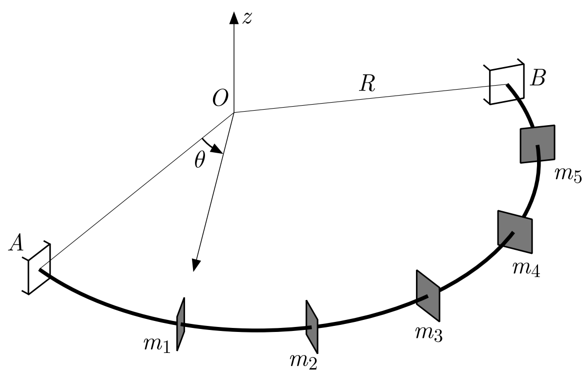

Fig. 3. Dynamic model of the curved beam

It is assumed that all the masses of curved beams are concentrated in discrete nodes. In this way, the translational mass m_i in the z direction of all nodes is the concentrated mass of the beam segment, and the mass of each arc segment is aggregated into a point mass in the node, and the distributed mass of each node is determined by the principle of statics. The mass coefficient related to rotation is given in the form of a moment of inertia about each axis. The moment of inertia of the section around axis ρ is represented by J_ρ and the moment of inertia of the section around circumferential axis θ is represented by J_θ. The lumped mass matrix of the curved beam structure containing the moment of inertia is obtained as

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & & & \\ & J_{\rho 1} & & & & & \\ & & J_{\theta 1} & & & & \\ & & & \ddots & & & \\ & & & & m_n & & \\ & & & & & J_{\rho n} & \\ & & & & & & J_{\theta n} \end{bmatrix} \tag{3.15}$$

4. Exemplary verification

The calculation example adopts q rectangular section constant curvature beam with section height of the intact structure $h = 15$ mm and width $b = 45$ mm. The area is $A = 675\,\text{mm}^2$, central angle $\varphi = 120°$, radius of curvature $R = 1$ m. Physical and geometric parameters are shown in Table 1.

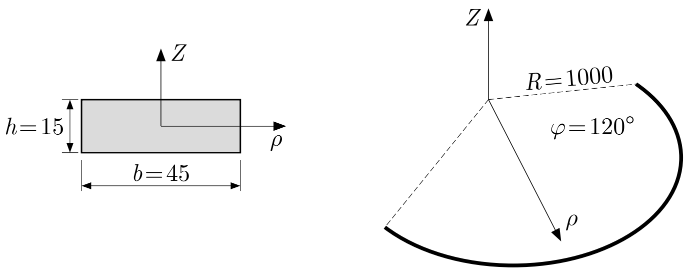

Fig. 4. Curved beam calculation diagram

Table 1. Physical and geometric parameters of the curved beam

Characteristic properties	Values
Young's modulus of elasticity E [MPa]	$2.06 \cdot 10^5$
Shear modulus of elasticity G [MPa]	$8.0 \cdot 10^4$
Inertia moment of area I [m^4]	$1.67 \cdot 10^{-5}$
Shear section coefficient k	1.2
Mass m [kg]	1803

4.1. Influence of damage on the natural frequency of the external curved beam

The frequency values of undamaged and damaged structures are found by the method presented in this paper. It is assumed that the damage location is in the middle of the span, and there are four damage coefficients ($\alpha = 0$, $\alpha = 0.16$, $\alpha = 0.33$, $\alpha = 0.5$). In order to verify the effectiveness of this method in analyzing free vibration problems of curved beams with different damage levels, a finite element model is established for the above problems and their numerical solutions are obtained. The frequency values obtained by the two methods are listed in Table 2. By comparing the results of the two methods, it can be seen that the larger the damage degree coefficient, the smaller the natural frequency. The theoretical value is consistent with the numerical analysis results. The error values of the two methods indicate that the frequency values obtained by applying the accurate algorithm under different damage degrees maintain good stable solutions. The correctness of the method for solving the out-of-plane vibration of damaged curved beams is verified.

Table 2. Natural frequency of the circular curved beam with damage

k	Frequency $\alpha = 0$		Frequency $\alpha = 0.16$		Frequency $\alpha = 0.33$		Frequency $\alpha = 0.5$	
	present	Ansys	present	Ansys	present	Ansys	present	Ansys
1	0.50579	0.5054	0.5103	0.5099	0.5117	0.5112	0.51189	0.5114
2	1.4396	1.4376	1.4639	1.4605	1.4729	1.4695	1.4764	1.4729
3	2.9581	2.9517	2.9688	2.9576	2.9728	2.9615	2.9745	2.9631
4	4.7408	4.7272	4.8855	4.8589	4.9445	4.9172	4.9694	4.9419
5	7.4276	7.3974	7.4575	7.3920	7.4701	7.4054	7.4755	7.4112

Figure 5 shows the results of the effect of the damage degree on the inherent vibration frequency of the externally curved beam. The order and the difference of the vertical coordinate relative frequency increases with the degree of damage, indicating that the inherent frequency decreases gradually with an increase of the damage factor α.

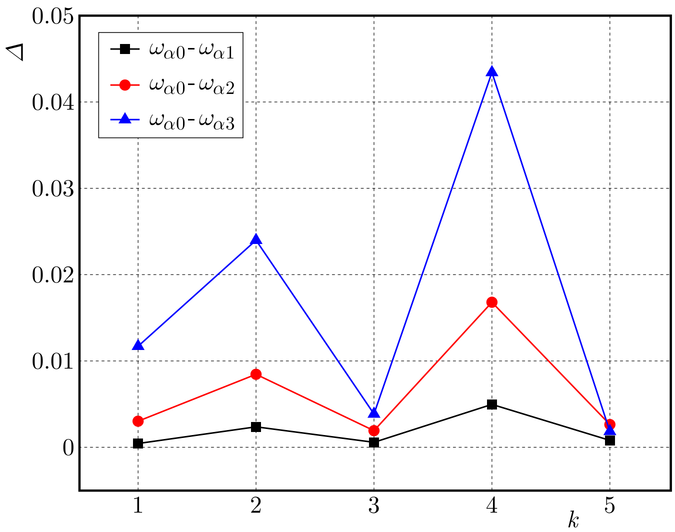

Fig. 5. Effect of the damage degree on the frequency

5. Conclusion

In this paper, a discrete dynamic model of a curved beam outside the plane is established. With the finite element model as a reference, an accurate solution can be obtained by using the proposed algorithm with fewer nodes, and has a high convergence. The paper also quantitatively analyzes the influence of damage on the natural frequency of the externally curved beam. The results show that the natural frequency of the externally curved beam decreases gradually with an increase of the damage degree. The above results show that the algorithm presented in this paper has good adaptability to solving damage characteristics of the externally curved beam and can provide a theoretical basis for the direct problem of damage identification of the externally curved beam.

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Manuscript received September 26, 2022; accepted for print March 4, 2023