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ANALYTICAL SOLUTION FOR ONE-DIMENSIONAL CONSOLIDATION OF UNSATURATED SOILS UNDER DYNAMIC LOAD

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Firstly, in this paper, based on the theory of the porous elastic medium and combined with the effective stress principle of unsaturated soil, a set of governing equations is established to describe consolidation of the unsaturated soil. Secondly, an analytical expression under any dynamic loads is obtained with the help of Laplace integral transformation. Finally, analysis of numerical examples under specific boundary conditions is made to discuss one--dimensional consolidation characteristics under harmonic loads and the influence of factors on the consolidation characteristics of unsaturated soil, such as excitation frequency and initial saturation.

Keywords: unsaturated soil, consolidation characteristics, dynamic load, analytical solution, numerical calculation

1. Introduction

Under an external load, the pore fluid is slowly discharged out of soil, and the process of gradual compression of soil is called soil consolidation. Consolidation characteristics of soil under complex conditions are of great significance to the actual construction process and operation stage of a project. For example, due to the effect of vehicle vibration load, long-term consolidation and deformation of the roadbed has a decisive impact on safety, durability, later operation and maintenance of the project in highway or railway engineering. Therefore, the ability to describe accurately the consolidation and deformation characteristics of soil is of great significance to present engineering.

At present, scholars have done a series of research on the problem of static and dynamic consolidation of the saturated soil foundation from different aspects (Pan *et al.*, 2006; Xie *et al.*, 2008, 2014; Toufigh and Ouria, 2009; Wang *et al.*, 2017a,b; Shi *et al.*, 2018). Compared with saturated soil, unsaturated soil not only contains the solid phase and liquid phase, but also contains a certain amount of the gas phase. It is widely present in arid, semi-arid areas and soil above the groundwater level. The research on consolidation of unsaturated soil began in the

1960s, when Fredlund and Rahardjo (1993) considered the continuity of pore water and pore gas, and established a relatively complete consolidation equation of unsaturated soil based on three-phase coupling characteristics. When discussing the problem of one-dimensional consolidation, the pore water dissipation equation is similar to the Terzaghi consolidation theory, but in the derivation process many assumptions that are inconsistent with the actual situation are introduced, such as constant total stress and so on. Since then, the analysis of the consolidation of unsaturated soil had achieved a series of results whether in terms of theoretical analysis (Sheng *et al.*, 2019; Qin *et al.*, 2010; Su and Xie, 2010; Ho *et al.*, 2014; Lo *et al.*, 2016; Wang *et al.*, 2017, 2018) or numerical simulation (Chen *et al.*, 1999; Yin and Ling, 2007; Huang *et al.*, 2009; Pedroso and Farias, 2011). However, in the research on the consolidation theory of unsaturated soil, most of the load forms imposed on the soil are dead loads or cyclic loads. In contrast, analytical solutions under the action of other dynamic loads are rarely reported in the literature. Based on the elastic theory of unsaturated porous media, this paper considers a function of three-phase coupling in unsaturated soil and establishes the consolidation equation under any dynamic loads. The Laplace integral transformation is used to finally obtain an explicit analytical solution for the problem of one-dimensional consolidation. Numerical examples under specific boundary conditions are used to discuss the law of influence of consolidation characteristics of the soil, frequency and saturation under harmonic loads.

2. Governing equation

Without considering the body force, the momentum balance equation of unsaturated porous media can be expressed as (Lo *et al.*, 2002)

$$R_{11}\Big(\frac{\partial \mathbf{u}_a}{\partial t}-\frac{\partial \mathbf{u}_s}{\partial t}\Big)=\theta_a\nabla P_a \qquad R_{22}\Big(\frac{\partial \mathbf{u}_w}{\partial t}-\frac{\partial \mathbf{u}_s}{\partial t}\Big)=\theta_w\nabla P_w \qquad \nabla\cdot\boldsymbol{\sigma}=\rho\ddot{\mathbf{u}}_s \tag{2.1}$$

where $\mathbf{u}_a$, $\mathbf{u}_w$ and $\mathbf{u}_s$ represent the displacement vectors of gas, water and solid soil skeletons, respectively; θ_a and θ_w represent the volume fractions of gas and water, respectively, and there are $\theta_a = S_a\varphi$, $\theta_w = S_w\varphi$, where S_a and S_w are the saturation of gas and water, respectively, φ is porosity of the soil; P_a and P_w are pore gas pressure and pore water pressure, respectively; $R_{11} = -[\theta_a^2\eta_1/k_s]k_{ra}$ is the viscous coupling coefficient of the solid phase and gas; $R_{22} = -[\theta_w^2\eta_2/k_s]k_{rw}$ is the viscous coupling coefficient of the solid phase and water, where η_1 and η_2 are viscosity coefficients of gas and water, respectively; k_s is the inherent permeability of porous media, k_{ra} and k_{rw} describe the relative permeability of gas and water, respectively; $\boldsymbol{\sigma}$ is the total stress tensor.

Unsaturated soil theory based on the suction stress between solid particles and the effective stress can be expressed as

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}'-(P_a+\sigma^s)\boldsymbol{\delta} \tag{2.2}$$

where $\boldsymbol{\sigma}'$ is the effective stress tensor; σ^s – suction stress between solid particles, which is a function related to the matrix suction and other factors, $\boldsymbol{\delta}$ – unit tensor. Regarding the expression of inter-particle attraction, domestic and foreign scholars have given different forms (Lu *et al.*, 2010; Chen *et al.*, 1993; Jiang *et al.*, 2004). According to current needs, this paper adopts the suction stress function with an explicit expression (Lu *et al.*, 2010), namely

$$\sigma^s=\begin{cases} P_a-P_w & \text{for} \quad P_a-P_w\leqslant 0\\ -S_e(P_a-P_w) & \text{for} \quad P_a-P_w\geqslant 0\end{cases} \tag{2.3}$$

where $S_e = (S_w - S_r)/(S_s - S_r)$ is the effective saturation, S_s – complete saturation, and S_r – residual saturation.

Combining the deformation of the solid framework and the pore fluid, the stress-strain relationship can be obtained (Lo *et al.*, 2014)

$$\boldsymbol{\sigma} = 2G\mathbf{e} + \Big[\Big(a_{11} - \frac{2}{3}G\Big)e + a_{12}\varepsilon_a + a_{13}\varepsilon_w\Big]\boldsymbol{\delta} \tag{2.4}$$

and

$$-\theta_a P_a = a_{12}e + a_{22}\varepsilon_a + a_{23}\varepsilon_w \qquad -\theta_w P_w = a_{13}e + a_{23}\varepsilon_a + a_{33}\varepsilon_w \tag{2.5}$$

where $\mathbf{e} = (\nabla\mathbf{u}_s + \nabla\mathbf{u}_s^{\mathrm{T}})/2$ is the strain tensor of the framework; $e = \nabla\mathbf{u}_s$ is the volumetric strain of the solid-phase framework; $\varepsilon_a = \nabla\mathbf{u}_a$ and $\varepsilon_w = \nabla\mathbf{u}_w$ are volumetric strains of gas and water, respectively; G is the shear modulus of the porous medium; $\boldsymbol{\sigma}$ is the unit tensor; a_{ij} $(i,j = 1,2,3)$ are coefficients of elasticity.

Combining Eqs. (2.2), (2.3) and (2.4), we can get

$$\boldsymbol{\sigma} = 2G\mathbf{e}+\Big[\Big(a_{11}-\frac{2}{3}G+A_1a_{12}+A_2a_{13}\Big)e+(a_{12}+A_1a_{22}+A_2a_{23})\varepsilon_a+(a_{13}+A_1a_{23}+A_2a_{33})\varepsilon_w\Big]\boldsymbol{\delta} \tag{2.6}$$

where $A_1 = (1 - S_e)/\theta_a$ and $A_2 = S_e/\theta_w$.

Through simultaneous Eqs. (2.5), the solution can be

$$\varepsilon_a = d_1e + d_2P_a + d_3P_w \qquad \varepsilon_w = d_4e + d_5P_a + d_6P_w \tag{2.7}$$

where

$$\begin{aligned} d_1 &= \frac{a_{12}a_{23} - a_{13}a_{23}}{a_{23}^2 - a_{22}a_{33}} & d_2 &= \frac{\theta_a a_{33}}{a_{23}^2 - a_{22}a_{33}} & d_3 &= \frac{-\theta_w a_{23}}{a_{23}^2 - a_{22}a_{33}} \\ d_4 &= \frac{a_{12}a_{22} - a_{12}a_{23}}{a_{23}^2 - a_{22}a_{33}} & d_5 &= \frac{-\theta_a a_{23}}{a_{23}^2 - a_{22}a_{33}} & d_6 &= \frac{\theta_w a_{22}}{a_{23}^2 - a_{22}a_{33}} \end{aligned}$$

Take divergence on both the left and right-hand sides of Eqs. (2.1), and substitute Eqs. (2.7) into it, then the coupled diffusion equation can be obtained

$$\begin{aligned} d_2\frac{\partial P_a}{\partial t} + d_3\frac{\partial P_w}{\partial t} + (d_1 - 1)\frac{\partial e}{\partial t} &= \frac{\theta_a}{R_{11}}\nabla^2 P_a \\ d_5\frac{\partial P_a}{\partial t} + d_6\frac{\partial P_w}{\partial t} + (d_4 - 1)\frac{\partial e}{\partial t} &= \frac{\theta_w}{R_{22}}\nabla^2 P_w \end{aligned} \tag{2.8}$$

Substitute Eqs. (2.7) into Eq. (2.6) to eliminate ε_a and ε_w, the total stress expression can be obtained as

$$\boldsymbol{\sigma} = 2\mathbf{G}e + (H_1e + H_2P_a + H_3P_w)\boldsymbol{\delta} \tag{2.9}$$

the coefficients in Eq. (2.9) are

$$\begin{aligned} H_1 &= a_{11} - \frac{2}{3}G + A_1a_{12} + A_2a_{13} + (a_{12} + A_1a_{22} + A_2a_{23})d_1 + (a_{13} + A_1a_{23} + A_2a_{33})d_4 \\ H_2 &= (a_{12} + A_1a_{22} + A_2a_{23})d_2 + (a_{13} + A_1a_{23} + A_2a_{33})d_5 \\ H_3 &= (a_{12} + A_1a_{22} + A_2a_{23})d_3 + (a_{13} + A_1a_{23} + A_2a_{33})d_6 \end{aligned}$$

Finally, by combining Eqs. $(2.1)_3$ and (2.9), a balance equation expressed by skeleton displacement, pore pressure and pore water pressure can be obtained

$$G\nabla^2\mathbf{u}_s + (H_1 + G)\nabla e + H_2\nabla P_a + H_3\nabla P_w = \rho\ddot{\mathbf{u}}_s \tag{2.10}$$

Based on the above derivation process, it is found that Eqs. (2.8) and (2.10) are three phase-coupling partial differential equations, which can be used to describe the consolidation of unsaturated soil.

3. Analytical solution of the one-dimensional dynamic consolidation problem

Consider the unsaturated soil layer as shown in Fig. 1, where thickness of the soil layer is h, and there is a dynamic load $f(t)$ acting on the surface, $e_{xx} = e_{yy} = 0$, $e = e_{zz} = \partial w/\partial z$.

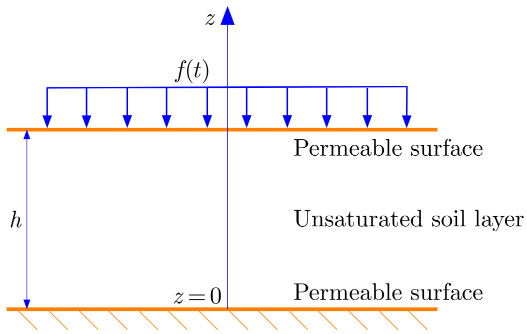

Fig. 1. Calculation model of unsaturated soil under the dynamic load $f(t)$

So control Eqs. (2.8) and (2.10) can be written as

$$\begin{aligned} q_1\frac{\partial P_a}{\partial t} + q_2\frac{\partial P_w}{\partial t} &= b_1\frac{\partial^2 P_a}{\partial z^2} + \gamma_1\frac{\partial f(t)}{\partial t} \\ q_3\frac{\partial P_a}{\partial t} + q_4\frac{\partial P_w}{\partial t} &= b_2\frac{\partial^2 P_w}{\partial z^2} + \gamma_2\frac{\partial f(t)}{\partial t} \end{aligned} \tag{3.1}$$

and

$$\frac{\partial w}{\partial z} = -\frac{f(t)}{2G+H_1} - \frac{H_2}{2G+H_1}P_a - \frac{H_3}{2G+H_1}P_w \tag{3.2}$$

where b_2/q_4 represents the diffusion coefficient of pore water pressure, called the consolidation coefficient, and is often expressed by the symbol c_v. The coefficients q_1, q_2, q_3, q_4, b_1, b_2, γ_1, γ_2 are respectively

$$\begin{aligned} q_1 &= d_2 - \frac{H_2(d_1-1)}{2G+H_1} & q_2 &= d_3 - \frac{H_3(d_1-1)}{2G+H_1} & q_3 &= d_5 - \frac{H_2(d_4-1)}{2G+H_1} \\ q_4 &= d_6 - \frac{H_3(d_4-1)}{2G+H_1} & b_1 &= \frac{\theta_a}{R_{11}} & b_2 &= \frac{\theta_w}{R_{22}} \\ \gamma_1 &= \frac{d_1-1}{2G+H_1} & \gamma_2 &= \frac{d_4-1}{2G+H_1} \end{aligned}$$

Since the exhaust and drainage cannot be completed at the moment of applying the dynamic load $f(t)$, it is considered that the content of water and gas remains the same. From Eqs. (3.1), we get

$$\int\limits_0^{0^+} b_1\frac{\partial^2 P_a}{\partial z^2}\,dt = \int\limits_0^{0^+} b_2\frac{\partial^2 P_w}{\partial z^2}\,dt = 0 \tag{3.3}$$

The formula: integral limit 0 and 0^+, respectively, represents the time before and after the load is applied.

From Eq. (3.2), we get

$$\frac{\partial w(z,0^+)}{\partial z} = -\frac{f(t)_{0^+}}{2G+H_1} - \frac{H_2}{2G+H_1}P_a - \frac{H_3}{2G+H_1}P_w \tag{3.4}$$

Combining Eqs. (3.3) and (2.8), we find

$$\begin{aligned}(d_1-1)\frac{\partial w(z,0^+)}{\partial z}+d_2P_a(z,0^+)+d_3P_w(z,0^+)=0\\(d_4-1)\frac{\partial w(z,0^+)}{\partial z}+d_5P_a(z,0^+)+d_6P_w(z,0^+)=0\end{aligned}\tag{3.5}$$

Substituting Eq. (3.4) into Eqs. (3.5), respectively, the initial conditions of the problem can be obtained as

$$P_a(z,0^+)=r_1f(t)_{0^+}\qquad P_w(z,0^+)=r_2f(t)_{0^+}\tag{3.6}$$

where

$$r_1=\frac{(1-d_4)q_2-(1-d_1)q_4}{(2G+H_1)(q_1q_4-q_2q_3)}\qquad r_2=\frac{(1-d_1)q_3-(1-d_4)q_1}{(2G+H_1)(q_1q_4-q_2q_3)}$$

Assume that the soil layer can both drain and exhaust on both sides at $z=0$ and $z=h$, and the boundary conditions are

$$P_a(0,t)=P_w(0,t)=0\qquad P_a(h,t)=P_w(h,t)=0\tag{3.7}$$

Without loss of generality, it is assumed that the pore gas pressure $P_a(z,t)$, pore water pressure $P_w(z,t)$ and the Fourier series form $\partial f(t)/\partial t$ are

$$P_a(z,t)=\sum_{n=0}^{\infty}P_{an}(t)\sin(\lambda_nz)\qquad P_w(z,t)=\sum_{n=0}^{\infty}P_{wn}(t)\sin(\lambda_nz)\tag{3.8}$$

and

$$\frac{\partial f(t)}{\partial t}=\sum_{m=0}^{\infty}B_m(t)\sin(\lambda_mz)\tag{3.9}$$

where $\lambda_m=m\pi/h$, P_{an}, P_{wn} and B_m are functions related to time t.

Combining Eqs. (3.8) and (3.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty}[q_1P'_{an}(t)+q_2P'_{wn}(t)+b_1\lambda_n^2P_{an}(t)]\sin(\lambda_nz)=\gamma_1\sum_{m=0}^{\infty}B_{am}(t)\sin(\lambda_mz)\\\sum_{n=0}^{\infty}[q_3P'_{an}(t)+q_3P'_{wn}(t)+b_2\lambda_n^2P_{wn}(t)]\sin(\lambda_nz)=\gamma_2\sum_{m=0}^{\infty}B_{wm}(t)\sin(\lambda_mz)\end{aligned}\tag{3.10}$$

Using the orthogonality of trigonometric functions and Eq. (3.1), we can see that the number of terms in m and n are equal, so that we obtain

$$\begin{aligned}q_1P'_{an}(t)+q_2P'_{wn}(t)+b_1\lambda_n^2P_{an}(t)=\gamma_1B_{an}(t)\\q_3P'_{an}(t)+q_4P'_{wn}(t)+b_2\lambda_n^2P_{wn}(t)=\gamma_2B_{wn}(t)\end{aligned}\tag{3.11}$$

Considering the initial conditions in Eqs. (3.6), integrating Eqs. (3.8) one arrives at

$$\begin{aligned}P_{an}(0)=\frac{2}{h\lambda_n}r_1f(t)_{0^+}[1-\cos(n\pi)]\\P_{wn}(0)=\frac{2}{h\lambda_n}r_2f(t)_{0^+}[1-\cos(n\pi)]\end{aligned}\tag{3.12}$$

Using Laplace integral transformation, Eqs. (3.11) can be written as follows

$$\begin{aligned}\begin{bmatrix}P_{an}(s)\\P_{wn}(s)\end{bmatrix}&=\frac{1}{X}\left(\begin{bmatrix}sq_4+b_2\lambda_n^2 & -sq_2\\ -sq_3 & sq_1+b_1\lambda_n^2\end{bmatrix}\begin{bmatrix}q_1 & q_2\\ q_3 & q_4\end{bmatrix}\begin{bmatrix}P_{an}(0)\\P_{wn}(0)\end{bmatrix}\right.\\&\left.\quad+\begin{bmatrix}sq_4+b_2\lambda_n^2 & -sq_2\\ -sq_3 & sq_1+b_1\lambda_n^2\end{bmatrix}\begin{bmatrix}\gamma_1B_{an}(s)\\ \gamma_2B_{wn}(s)\end{bmatrix}\right)\end{aligned} \tag{3.13}$$

where $X=\varDelta s^2+\lambda_n^2(b_1q_4+b_2q_1)s+b_1b_2\lambda_n^4$ and $\varDelta=q_1q_4-q_2q_3$.

Solving Eq. (3.13), we get

$$\begin{aligned}P_{an}(s)&=\frac{s}{s^2+\beta_n^2}P_{an}(0)+\frac{b_2\lambda_n^2q_1}{\beta_n\varDelta}\frac{\beta_n}{s^2+\beta_n^2}P_{an}(0)+\frac{b_2q_2\lambda_n^2}{\varDelta\beta_n}\frac{\beta_n}{s^2+\beta_n^2}P_{wn}(0)\\&\quad+\frac{q_4\gamma_1}{\varDelta}\frac{s}{s^2+\beta_n^2}B_{an}(s)+\frac{b_2\lambda_n^2\gamma_1}{\beta_n\varDelta}\frac{\beta_n}{s^2+\beta_n^2}B_{an}(s)-\frac{q_2\gamma_2}{\varDelta}\frac{s}{s^2+\beta_n^2}B_{wn}(s)\\P_{wn}(s)&=\frac{b_1q_3\lambda_n^2}{\varDelta\beta_n}\frac{\beta_n}{s^2+\beta_n^2}P_{an}(0)+\frac{s}{s^2+\beta_n^2}P_{wn}(0)+\frac{b_1\lambda_n^2q_4}{\beta_n\varDelta}\frac{\beta_n}{s^2+\beta_n^2}P_{wn}(0)\\&\quad-\frac{\gamma_1q_3}{\varDelta}\frac{s}{s^2+\beta_n^2}B_{an}(s)+\frac{q_1\gamma_2}{\varDelta}\frac{s}{s^2+\beta_n^2}B_{wn}(s)+\frac{\gamma_2b_1\lambda_n^2}{\varDelta\beta_n}\frac{\beta_n}{s^2+\beta_n^2}B_{wn}(s)\end{aligned} \tag{3.14}$$

where $\beta_n^2=(1/\varDelta)\lambda_n^2(b_1q_4+b_2q_1)s+(1/\varDelta)b_1b_2\lambda_n^4$.

Using Laplace inverse transformation, Eqs. (3.14) can be expressed in the time domain

$$\begin{aligned}P_{an}(t)&=P_{an}(0)\cos(\beta_nt)+\frac{b_2q_1\lambda_n^2}{\beta_n\varDelta}P_{an}(0)\sin(\beta_nt)+\frac{b_2q_2\lambda_n^2}{\varDelta\beta_n}P_{wn}(0)\sin(\beta_nt)\\&\quad+\frac{q_4\gamma_1}{\varDelta}\cos(\beta_nt)B_{an}(t)+\frac{b_2\gamma_1\lambda_n^2}{\beta_n\varDelta}\sin(\beta_nt)B_{an}(t)-\frac{q_2\gamma_2}{\varDelta}\cos(\beta_nt)B_{wn}(t)\\P_{wn}(t)&=\frac{b_1q_3\lambda_n^2}{\varDelta\beta_n}P_{an}(0)\sin(\beta_nt)+P_{wn}(0)\cos(\beta_nt)+\frac{b_1q_4\lambda_n^2}{\beta_n\varDelta}P_{wn}(0)\sin(\beta_nt)\\&\quad-\frac{q_3\gamma_1}{\varDelta}\cos(\beta_nt)B_{an}(t)+\frac{q_1\gamma_2}{\varDelta}\cos(\beta_nt)B_{wn}(t)+\frac{b_1\lambda_n^2\gamma_2}{\beta_n\varDelta}\sin(\beta_nt)B_{wn}(t)\end{aligned} \tag{3.15}$$

Combining Eqs. (3.2), (3.8) and (3.15), the settlement expression of the unsaturated soil layer under any dynamic load can be finally obtained

$$\begin{aligned}S(t)&=-\int\limits_0^h\frac{\partial w}{\partial z}\,dz=\frac{H_2}{2G+H_1}\sum_{n=0}^{\infty}\frac{1-\cos(n\pi)}{\lambda_n}P_{an}(t)+\frac{f(t)h}{2G+H_1}\\&\quad+\frac{H_3}{2G+H_1}\sum_{n=0}^{\infty}\frac{1-\cos(n\pi)}{\lambda_n}P_{wn}(t)\end{aligned} \tag{3.16}$$

4. Numerical example analysis

As a numerical example, consider the following simple harmonic load on the surface of the soil layer

$$f(t)=\frac{P^*}{2}[1+\cos(\omega t)] \tag{4.1}$$

where $P^*/2$ and ω, respectively, represent the amplitude and frequency of the harmonic load.

Then the analytical expression of settlement deformation is

$$\begin{aligned} S(t) &= -\int\limits_0^h \frac{\partial w}{\partial z}\,dz = \frac{H_2}{2G+H_1}\sum_{n=0}^{\infty}\frac{1-\cos(n\pi)}{\lambda_n}P_{an}(t) + \frac{P^* h[1+\cos(\omega t)]}{2(2G+H_1)} \\ &+ \frac{H_3}{2G+H_1}\sum_{n=0}^{\infty}\frac{1-\cos(n\pi)}{\lambda_n}P_{wn}(t) \end{aligned} \tag{4.2}$$

The specific expressions of each variable in Eq. (4.2) are shown in Appendix A.

The one-dimensional consolidation equation of unsaturated soil considering the pore liquid phase and pore gas phase and their mutual coupling effects is derived and solved by the Laplace transform. Numerical analysis is carried out on the present calculation example under the action of a simple harmonic load. The law of influence of the excitation frequency, initial saturation and depth on the one-dimensional consolidation deformation and characteristics of unsaturated soil is discussed. Among them, physical characteristic parameters of unsaturated soil used in the numerical examples, such as elastic coefficient, matrix suction and permeability coefficient of the fluid phase are all related to saturated soil. Therefore, the expressions of specific parameters are detailed in Appendix B. When considering a one-dimensional unsaturated clay layer, the calculated parameter values are shown in Table 1.

Table 1. Basic physical parameters of clay

Parameter type	Symbol	Numerical value
Porosity	φ	0.475
Fitting parameter	ζ	$1.168\,\mathrm{m}^{-1}$
Fitting parameter	n	1.165
Inherent permeability	k_s	$1.699\cdot 10^{-14}\,\mathrm{m}^2$
Shear modulus	G	$2.4\cdot 10^{6}$ Pa
Gas bulk modulus	K_1	$1.45\cdot 10^{5}$ Pa
Bulk modulus of water	K_2	$2.25\cdot 10^{9}$ Pa
Solid bulk modulus	K_s	$3.5\cdot 10^{10}$ GPa
Consolidation bulk modulus	K_b	$4.5\cdot 10^{6}$ Pa
Density of water	ρ_w	$997\,\mathrm{kg/m}^3$
Coefficient viscosity of gas	η_1	$18\cdot 10^{-6}$ Pa·s
Coefficient viscosity of water	η_2	0.001 Pa·s
Acceleration of gravity	g	$9.8\,\mathrm{m/s}^2$
Pore connectivity parameter	η	1.165

Figure 2 shows the influence of the excitation frequency on deformation under different initial saturations. Figures 2a,b,c and 2d are change curves of settlement under different excitation frequencies when the initial saturation S_w is 0.65, 0.75, 0.85 and 1,00, respectively. In addition, dimensionless depth, dimensionless pore water pressure and dimensionless time are defined as z/h and $T = (c_v/h^2)t$. Firstly, the settlement curves under different saturations are discussed separately, and it is found that when the excitation frequency is 0 Hz the settlements show a steady development, which is shown as a consolidation curve under a constant load. When the excitation frequencies are 0.1 Hz and 10 Hz, the settlement curves show obvious dynamic load characteristics, and with an increases of the excitation frequency, the transient response frequency of the settlement also accelerates.

By comparing the calculation results of the initial saturation S_w of 0.65, 0.75 and 0.85 in the unsaturated state with the calculation results in the saturated state ($S_w = 1.00$), it is found that when the excitation frequency $\omega = 0$ Hz, the time for saturated soil clay to reach consolidation

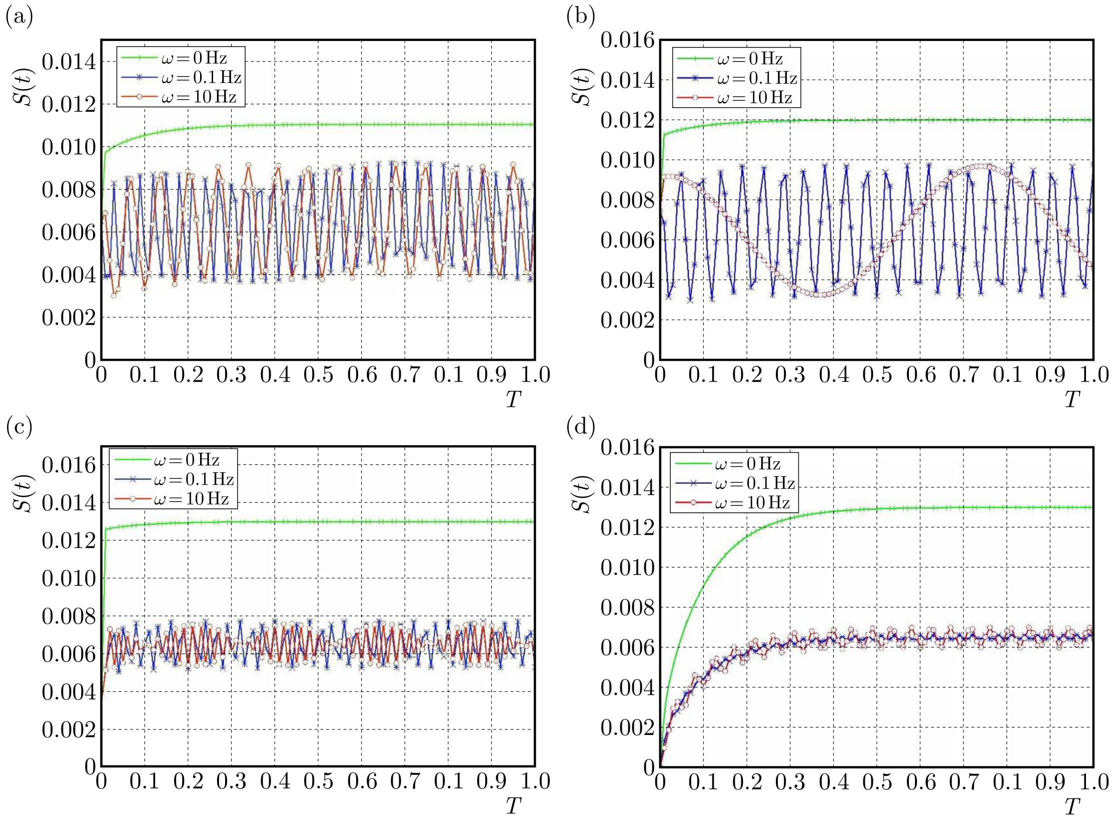

Fig. 2. Influence curves of the frequency on deformation under different saturations: (a) $S_w = 0.65$, (b) $S_w = 0.75$, (c) $S_w = 0.85$, (d) $S_w = 1.00$

stability is significantly longer than that for unsaturated soil clay. When the excitation frequency ω is 0.1 Hz and 10 Hz, the amount of sedimentation at a certain moment decreases with an increase of the initial saturation. In addition, the transient response lasts longer in saturated clay, while the initial saturation is higher, the transient response will disappear more quickly in unsaturated clay. All in all, the value of settlement under the constant load ($\omega = 0$) is always greater than that under the cyclic load ($\omega > 0$). However, for the settlement under the dynamic load it is difficult to maintain stability during the load-bearing period, and the effect of saturation on consolidation deformation is also significantly far from the load frequency.

Figure 3 shows the change curve of pore water pressure P_w/P^* along the soil depth z/h when the dimensionless time T is 10^{-5}, 10^{-3} and 10^{-1}, respectively. Figures 3a,b,c and 3d show the distribution curve when the initial saturation is 0.65, 0.75, 0.85 and 1.00, respectively. In addition, the dimensionless frequency is $\Omega = \omega h^2/c_v$, and the dimensionless frequency is calculated $\Omega = 10^7$ according to specific conditions.

It can be observed from Fig. 3 that under the unsaturated condition, the pore water pressure in the area near the top and bottom of the soil layer has different degrees of oscillation, but this situation does not appear in the saturated state. The reason for this phenomenon is that the frequency of the applied dynamic load is different for different saturation, which causes a change of pore water pressure at the boundary. At the dimensionless frequency $\Omega = 10^7$, when the saturation S_w is 0.65, 0.75 and 0.85, the corresponding excitation frequencies ω are 0.013 Hz, 0.042 Hz and 0.114 Hz, respectively. When the soil is saturated, the corresponding excitation frequency ω is 857 Hz. Therefore, given a dimensionless frequency due to the difference in soil saturation makes the excitation frequency vary greatly. This phenomenon strongly depends on the size of the excitation frequency.

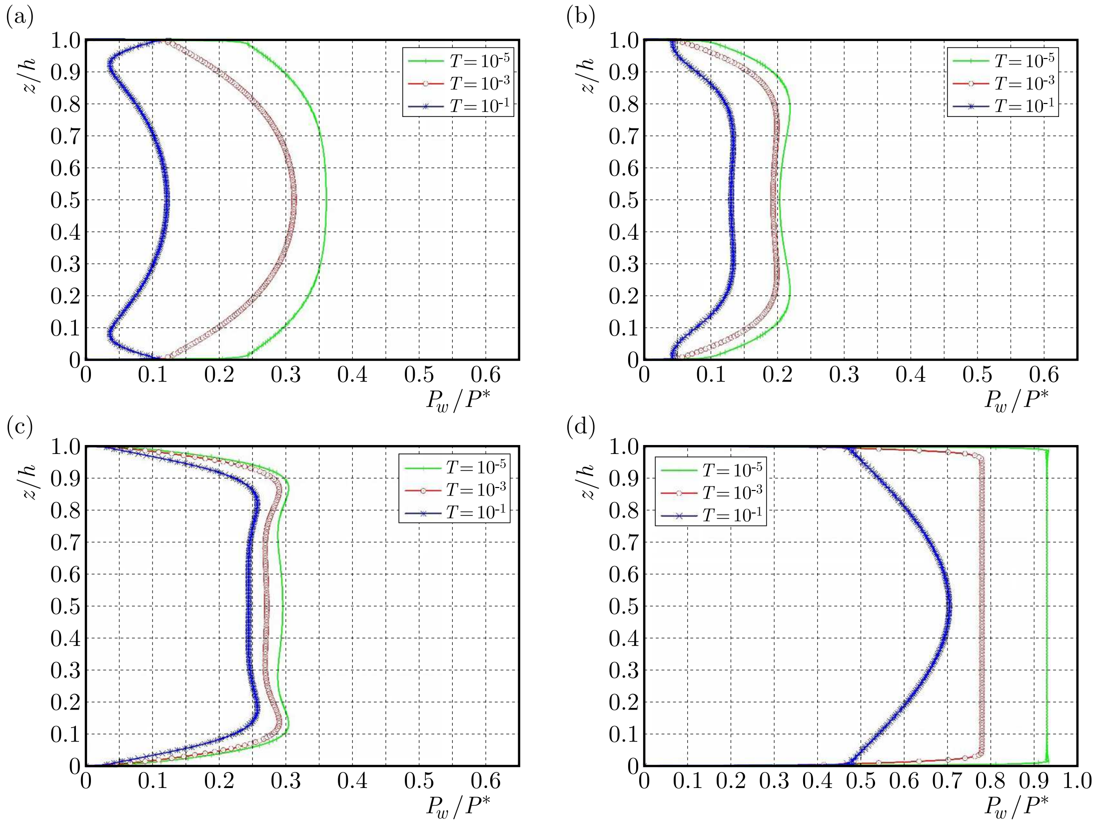

Fig. 3. Distribution law of pore water pressure and depth under complex conditions: (a) $S_w = 0.65$, (b) $S_w = 0.75$, (c) $S_w = 0.85$, (d) $S_w = 1.00$

5. Conclusion

Based on the theory of porous elastic media, an effective stress formula of unsaturated soil expressed by suction stress is adopted, and a analytical solution of one-dimensional consolidation of unsaturated soil under dynamic load is obtained by the Laplace transformation. In addition, specific numerical analysis and discussion of one-dimensional consolidation behavior of unsaturated clay under specific boundary conditions of a simple harmonic load are carried out. It is clear that the excitation frequency of the simple harmonic load and the initial saturation of the soil affects consolidation settlement. The results show that the settlement of soil under the constant load is always greater than the settlement under the dynamic load. For the dynamic load, the settlement at a certain moment decreases with an increase of saturation, and the transient response under the dynamic load is also closely related to the load frequency and saturation. Moreover, pore water pressure exhibits an obvious oscillation phenomenon in the area near the drainage boundary and an oscillatory trend in the unsaturated soil under the dynamic load.

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Appendix A

The Fourier coefficient of a simple harmonic load Eq. (4.1) is

$$B_m(t) = \frac{P^*\omega[\cos(m\pi) - 1]}{m\pi}\sin(\omega t) = Y_m \sin(\omega t) \tag{A.1}$$

Combine Eq. (3.15) to get

$$\begin{aligned} P_{an}(t) &= (D_1 + G_1)\mathrm{e}^{-C_1 t} + (D_2 + G_2)\mathrm{e}^{-C_2 t} + G_5\cos(\omega t) + G_6\sin(\omega t) \\ P_{wn}(t) &= (D_3 + G_3)\mathrm{e}^{-C_1 t} + (D_4 + G_4)\mathrm{e}^{-C_2 t} + G_7\cos(\omega t) + G_8\sin(\omega t) \end{aligned} \tag{A.2}$$

The coefficients in Eqs. (A.2) are

$$\begin{aligned}
C_1 &= \frac{\lambda_n^2}{2}\left(\frac{q_1 b_2 + q_4 b_1}{\varDelta} + \sqrt{\frac{1}{\varDelta} + (q_1 b_2 + q_4 b_1)^2 - \frac{4 b_1 b_2}{\varDelta}}\right) \\
C_2 &= \frac{\lambda_n^2}{2}\left(\frac{q_1 b_2 + q_4 b_1}{\varDelta} - \sqrt{\frac{1}{\varDelta} + (q_1 b_2 + q_4 b_1)^2 - \frac{4 b_1 b_2}{\varDelta}}\right) \\
D_1 &= \frac{1}{C_1 - C_2}\Big[P_{an}(0)\Big(C_1 - \frac{q_1 b_2 \lambda_n^2}{\varDelta}\Big) - \frac{q_2 b_2 \lambda_n^2}{\varDelta}P_{wn}(0)\Big] \\
D_2 &= \frac{1}{C_1 - C_2}\Big[P_{an}(0)\Big(\frac{q_1 b_2 \lambda_n^2}{\varDelta} - C_2\Big) + \frac{q_2 b_2 \lambda_n^2}{\varDelta}P_{wn}(0)\Big] \\
D_3 &= \frac{1}{C_1 - C_2}\Big[P_{wn}(0)\Big(C_1 - \frac{q_4 b_1 \lambda_n^2}{\varDelta}\Big) - \frac{q_3 b_1 \lambda_n^2}{\varDelta}P_{an}(0)\Big] \\
D_4 &= \frac{1}{C_1 - C_2}\Big[P_{wn}(0)\Big(\frac{q_4 b_1 \lambda_n^2}{\varDelta} - C_2\Big) + \frac{q_3 b_1 \lambda_n^2}{\varDelta}P_{an}(0)\Big] \\
G_1 &= \frac{\omega M_2 - C_1 M_1}{(C_2 - C_1)(C_1^2 + \omega^2)} \qquad G_2 = \frac{C_2 M_1 - \omega M_2}{(C_2 - C_1)(C_1^2 + \omega^2)} \\
G_3 &= \frac{\omega M_4 - C_1 M_3}{(C_2 - C_1)(C_1^2 + \omega^2)} \qquad G_4 = \frac{C_2 M_3 - \omega M_4}{(C_2 - C_1)(C_1^2 + \omega^2)} \\
G_5 &= \frac{M_1(C_1 C_2 - \omega^2) - \omega M_2(C_1 + C_2)}{(C_1^2 + \omega^2)(C_2^2 + \omega^2)} \qquad G_6 = \frac{M_1\omega(C_1 + C_2) + M_2(C_1 C_2 - \omega^2)}{(C_1^2 + \omega^2)(C_2^2 + \omega^2)} \\
G_7 &= \frac{M_3(C_1 C_2 - \omega^2) - \omega M_4(C_1 + C_2)}{(C_1^2 + \omega^2)(C_2^2 + \omega^2)} \qquad G_8 = \frac{M_3\omega(C_1 + C_2) + M_4(C_1 C_2 - \omega^2)}{(C_1^2 + \omega^2)(C_2^2 + \omega^2)}
\end{aligned}$$

where

$$M_1 = \frac{(q_4 Y_{an} - q_2 Y_{wn})\omega}{\varDelta} \qquad M_2 = \frac{b_2 \lambda_n^2 Y_{an}}{\varDelta} \qquad M_3 = \frac{(q_1 Y_{wn} - q_3 Y_{an})\omega}{\varDelta}$$

Appendix B

In the model of unsaturated soil, the relationship among physical, mechanical parameters and the saturation is as follows.

Coefficient of linear elasticity (Lo *et al.*, 2005)

$$
\begin{aligned}
a_{11} &= \frac{K_s}{N_3}\{(1-\varphi)N_1[K_1K_2 + K_1N_2S_a + K_2N_2(1-S_a)] \\
&\quad + K_bK_s\varphi[K_1(1-S_a) + K_2S_1 + N_2]\} \\
a_{12} = a_{21} &= \frac{N_1K_sK_1S_a(K_2+N_2)}{N_3} \qquad a_{13} = a_{31} = \frac{N_1K_s\varphi K_2(1-S_a)(K_1+N_2)}{N_3} \\
a_{22} &= \frac{K_1S_a^2\varphi}{N_3}\Big[K_s^2\varphi\Big(K_2 + \frac{N_2}{S_a}\Big) + \frac{K_2N_1N_2}{S_a}(1-S_a)\Big] \\
a_{23} = a_{32} &= -\frac{K_1K_2\varphi S_a(1-S_a)(N_1N_2 - \varphi K_s)}{N_3} \\
a_{33} &= \frac{K_2(1-S_a)\varphi}{N_3}\Big[K_s^2\varphi\Big(K_1 + \frac{N_2}{1-S_a}\Big) + \frac{K_1N_1N_2S_a}{1-S_a}\Big]
\end{aligned}
$$

where K_b is the bulk modulus of the soil skeleton; K_1, K_2 and K_s respectively represent the bulk modulus of air, water and solid soil particles. Among them, the parameters N_1, N_2 and N_3 are defined as

$$
\begin{aligned}
&N_1 = K_s(1-\varphi) - K_b \qquad N_2 = \frac{dp_c}{dS_a}S_a(1-S_a) \\
&N_3 \equiv N_1[K_1N_2S_a + K_1K_2 + K_2N_2(1-S_a)] + K_s^2[K_1(1-S_a) + N_2 + K_2S_a]
\end{aligned}
\tag{B.1}
$$

where p_c is the suction matrix. In the V-G model (Genuchten, 1980)

$$
S_e = [1 + (\zeta h_c)^n]^{-m} \tag{B.2}
$$

where n, m and ζ are fitting parameters, $m = 1 - (1/n)$.

Combining Eq. (B.2) the expression for N_2 can be obtained

$$
N_2 = \frac{\rho_w g}{mn\zeta}\frac{S_aS_w}{S_s - S_r}\Big[\Big(\frac{1-S_r}{S_s-S_r} - \frac{S_a}{S_s-S_r}\Big)^{-1/m} - 1\Big]^{(1/n)-1}\Big(\frac{1-S_r}{S_s-S_r} - \frac{S_a}{S_s-S_r}\Big)^{-1-(1/m)} \tag{B.3}
$$

The relative permeability of pore gas and water (Wang *et al.*, 2017) is

$$
k_{ra} = (1-S_e)\Big(1 - \sqrt[m]{S_e}\Big)^{2m} \qquad k_{ra} = S_e^{\eta}\Big[1 - \Big(1 - \sqrt[m]{S_e}\Big)^m\Big]^2 \tag{B.4}
$$

where η represents a parameter related to pore connectivity.

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