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COMPARISON OF LONG TIME SIMULATION OF HAMILTON AND LAGRANGE GEOMETRY DYNAMICAL MODELS OF A MULTIBODY SYSTEM

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The geometry dynamical modeling method for a double pendulum is explored with the Lie group and a double spherical space method. Four types of Lagrange equations are built for relative and absolute motion with the above two geometry methods, which are then used to explore the influence of different expressions for motion on the dynamic modeling and computations. With Legendre transformation, the Lagrange equations are transformed to Hamilton ones which are dynamical models greatly reduced. The models are solved by the same numerical method. The simulation results show that they are better for the relative group than for the absolute one in long time simulation with the same numerical computations. The Lie group based result is better than the double spherical space one.

Keywords: double pendulum, Lie group, double spherical space, Lagrange, Hamilton equations

1. Introduction

The geometry method can conserve a geometrical structure and give a simple mathematical expression for a dynamical model, which is significant to dynamical modeling of multibody systems. As a mathematical concept, the geometry theory is very complex to the common engineer, so the way of using it in the multibody dynamics modeling is very important. From the mathematical derivation, the geometry method uses the attitude matrix or the attitude vector as the modeling element, which can express an obvious reduction in the dynamical modeling and avoidance of the complex triangle transformation. The numerical method which is based on the geometry dynamical model can also avoid the triangle computations, which can enhance the numerical efficiency and make the system have long time simulation stability. So the exploration of the geometry dynamical modeling of a multibody system is significant to the accuracy, efficiency and stability in the multibody systems modeling.

In recent years, the geometry dynamical modeling method for multibody systems has been researched extensively. Lee *et al.* (2009) explored the geometry modeling in double spherical space with a spherical pendulum and a plastic rod. Ding and Pan (2014) explored a high order variational integrator with constraints of a multibody system using the spherical pendulum as an example. Müller (2014, 2021, 2022) and Müller and Terze (2014) constructed a multibody system dynamical model of a parallel mechanism, and combined the topology method and the Lie group theory. He also explored the relation of the Lie group structure with constraints of a time integrated multibody system, and found a close loop expression in form of a high order differention equation of motion of the system joints. With the Lie group integrator and Cayley transformation, an implicit generalized α method is presented to enhance the simulation efficiency. Bjorkenstam *et al.* (2018) explored the discrete geometry method for a multibody system with constraint multipliers, which was used in the optimal control. Celledoni *et al.* (2021) built a Lie group integrator for a chain system and studied it numerically. Holzinger and Gestmayr (2021) used the Lie group time integrator in consistency computations of the rotation vector or Euler angles to avoid the singularity and enhance the accuracy. This method has also an advantage of transportability. Rong *et al.* (2020) explored dynamical equations of geometrically accurate thin-walled beams with an arbitrary sectional base on the local coordinate of $SE(3)$ applying the Lie group generalized α integrate method. Ding *et al.* (2019) explored the constraint stability equation with Lie group theory for a 3D rigid body and double pendulum in space. An implicit solving method for the Lagrange equation and constraint Hamilton equation was built. This method can conserve stability of long time simulation of the displacement, velocity and acceleration. Urkullu *et al.* (2019) explored the direct central difference method for a solving multibody system, so that the equation could be solved directly without the of reducing the order. Terze *et al.* (2015) analyzed the solving method of the first kind differential-algebra equation. With the least square constraint stability mapping method eliminating the constraint break problem of generalized coordinate and velocity in integration, the method was testified by the heavy top and the satellite model. Arnold and Brüls (2015) explored convergence of the Lie group by coupling first order error recursion method of the differential-algebra solution. This research indicates that transient vibration can be eliminated by perturbation of the initial value or parameter simplification. Sonneville and Brüls (2014) used a directly differential method and an adjoint variable method to estimate semi-analytical sensitivity. The sensitivity of a multibody system under Lie group expression was also explored.

In this study, the geometry dynamical model of a double pendulum with its own inertia is constructed. The dynamical model is based on two different geometry methods – the Lie group and attitude vector, and two different motions – the absolute and relative movement. Four types of dynamical models are built, and their characteristics are analyzed according to numerical computations. This research discloses foundations of the geometry method used in multibody systems more deeply.

2. Geometry method for a multibody system

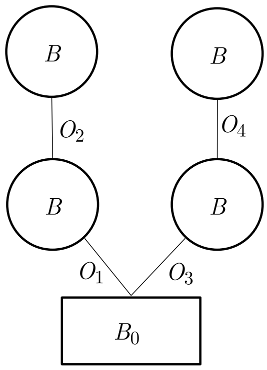

Fig. 1. The tree structure of a multibody system

Figure 1 shows a tree structure of a multibody system. B_i represents the rigid body, O_i represents the joint of bodies. Here, the joints are plane. Supposing that the rotation angle of the joint is θ_i, so the rotation matrix is $\mathbf{R}_i$. If the relative angle is used to express rotation of each joints, the attitude matrix of each rigid body has a recursive relation. Using B_0-B_1-B_2 as the example, supposing that B_0 is the root of the tree, the attitude matrix of B_1 and B_2 are expressed as

$$\mathbf{R}_{B1} = \mathbf{R}_1 \qquad \mathbf{R}_{B2} = \mathbf{R}_1\mathbf{R}_2 \tag{2.1}$$

Differentiate (2.1) to obtain the relation between the angular velocity and attitude matrix

$$\dot{\mathbf{R}}_{B1} = \mathbf{R}_1\mathbf{S}(\omega_1) \qquad \dot{\mathbf{R}}_{B2} = \mathbf{R}_1\mathbf{S}(\omega_1)\mathbf{R}_2 + \mathbf{R}_1\mathbf{R}_2\mathbf{S}(\omega_2) \tag{2.2}$$

In (2.2), $\mathbf{S}(\cdot)$ is the skew symmetric matrix. For the plane rotation joint, the angular velocity ω can be extracted and (2.2) can be transformed as

$$\dot{\mathbf{R}}_{B1} = \omega_1\mathbf{R}_1\mathbf{S}(1) \qquad \dot{\mathbf{R}}_{B2} = \omega_1\mathbf{R}_1\mathbf{S}(1)\mathbf{R}_2 + \omega_2\mathbf{R}_1\mathbf{R}_2\mathbf{S}(1) \tag{2.3}$$

The skew matrix is $\mathbf{S}(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. According to (2.1), (2.2) and (2.3), $\mathbf{R}_1$, $\mathbf{R}_2$ represent the relative attitude of B_0-B_1 and B_1-B_2, respectively. $\mathbf{R}_{B1}$, $\mathbf{R}_{B2}$ are absolute attitude matrices of an arbitrary rigid body relative to the root B_0. The absolute attitude matrix and absolute angular velocity satisfy the relation which is similar to the relative expressions as

$$\dot{\mathbf{R}}_{B1} = \mathbf{R}_{B1}\mathbf{S}(\omega_{B1}) \qquad \dot{\mathbf{R}}_{B2} = \mathbf{R}_{B2}\mathbf{S}(\omega_{B2}) \tag{2.4}$$

Comparing the second term of equation (2.3) and (2.4), these two terms must be equal. It can be obtained by transformation of (2.3). The attitude matrix can be expressed as the triangle one

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.5}$$

Substituting (2.5) into (2.3), the relationship angular and absolute velocity is

$$\omega_{B2} = \omega_1 + \omega_2 \tag{2.6}$$

The term $\mathbf{R}_1\mathbf{S}(1)\mathbf{R}_2$ can be written as

$$\mathbf{R}_1\mathbf{S}(1)\mathbf{R}_2 = \mathbf{R}_1\mathbf{R}_2\mathbf{S}(1) \tag{2.7}$$

Equation (2.6) indicates that in the planar multibody system, the absolute angular velocity of each body is equal to the sum of the angular velocities of every body on the chain to the root of the tree. It also conforms to the universal law of planar rigid body motion. So with the Lie group theory, the attitude of the rigid body is expressed by a matrix. The attitude of the rigid body can also be expressed as a vector. Supposing that the attitude vector of the rigid body is $\mathbf{q}$, then the triangle expression of it is

$$\mathbf{q} = [\cos\theta, \sin\theta] \tag{2.8}$$

Differentiate (2.8) to obtain the relation between the angular velocity and attitude vector

$$\dot{\mathbf{q}} = \omega[-\sin\theta, \cos\theta] = \omega\mathbf{S}_1\mathbf{q} \tag{2.9}$$

Similar to equation (2.1), the attitude of each rigid body in the tree structure needs to be expressed by the attitude vector, which is the basis for dynamical analysis. The attitude of B_1 relative to B_0 can be written as $\mathbf{q}_{B1} = \mathbf{q}_1$, and the absolute attitude vector of B_2 relative to B_0 is

$$\mathbf{q}_{B2} = \mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_1 + \mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_2 \tag{2.10}$$

where $\mathbf{T}_1$, $\mathbf{T}_2$ in (2.10) are as follows. Equation (2.10) can be testified by triangle transformations. Differentiate (2.10) to obtain the relation for the angular velocity and attitude vector

$$\dot{\mathbf{q}}_{B2} = -\omega_1 \mathbf{q}_1^{\mathrm{T}} \mathbf{S}_1 \mathbf{T}_1 \mathbf{q}_2 \mathbf{e}_1 + \omega_2 \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_1 \mathbf{S}_1 \mathbf{q}_2 \mathbf{e}_1 - \omega_1 \mathbf{q}_1^{\mathrm{T}} \mathbf{S}_1 \mathbf{T}_2 \mathbf{q}_2 \mathbf{e}_2 + \omega_2 \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_2 \mathbf{S}_1 \mathbf{q}_2 \mathbf{e}_2 \tag{2.11}$$

With $\mathbf{S}_1\mathbf{T}_1 = \mathbf{T}_2$, $\mathbf{T}_1\mathbf{S}_1 = -\mathbf{T}_2$, $\mathbf{S}_1\mathbf{T}_2 = -\mathbf{T}_1$, $\mathbf{T}_2\mathbf{S}_1 = \mathbf{T}_1$, equation (2.11) can be transformed to

$$\dot{\mathbf{q}}_{B2} = (\omega_1 + \omega_2)(\mathbf{q}_1^{\mathrm{T}} \mathbf{T}_1 \mathbf{q}_2 \mathbf{e}_2 - \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_2 \mathbf{q}_2 \mathbf{e}_1) = (\omega_1 + \omega_2) \mathbf{S}_1 \mathbf{q}_B \tag{2.12}$$

According to equation (2.12), the angular velocity of the rigid body based on the attitude vector also satisfies the cumulative relation.

3. Dynamical modeling with relative rotation matrix and vector

In this part, the double pendulum geometry dynamical model is built by the relative attitude matrix and vector. The double pendulum is shown in Fig. 2. Suppose that the rotation matrix

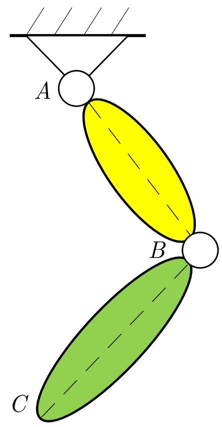

Fig. 2. Sketch map of the double pendulum

of AB is $\mathbf{R}_1$, which expresses the rotation relative to the ground. So it is also the absolute rotation matrix which satisfies $\mathbf{R}_A = \mathbf{R}_1$. The rotation matrix of BC relative to AB is $\mathbf{R}_2$, so the absolute rotation matrix of BC is $\mathbf{R}_B = \mathbf{R}_1\mathbf{R}_2$. Supposing the mass center position vectors of AB and BC are $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, respectively, the angular velocity of AB is ω_1, the angular velocity of BC relative to AB is ω_2, so the absolute angular velocity of BC is

$$\dot{\mathbf{R}}_B = \mathbf{R}_B \mathbf{S}(\omega_1 + \omega_2) \tag{3.1}$$

In Eq. (3.1), $\mathbf{R}_1\mathbf{S}_1\mathbf{R}_2 = \mathbf{R}_1\mathbf{R}_2\mathbf{S}_1$. The rotation inertia of AB and BC along the mass center are J_1 and J_2, respectively. The position of B in AB is $\mathbf{q}_2$, the velocity of AB mass center is $\boldsymbol{\rho}_1\omega_1$, the kinetic and potential energy are

$$\mathbf{T}_1 = \frac{1}{2}(J_1 + m_1 \boldsymbol{\rho}_1^{\mathrm{T}} \boldsymbol{\rho}_1)\omega_1^2 \qquad V_1 = -m_1 g \mathbf{e}_2^{\mathrm{T}} \mathbf{R}_1 \boldsymbol{\rho}_1 \tag{3.2}$$

According to the rotation relation, the displacement of mass center of BC is $\mathbf{R}_1\mathbf{q}_1 + \mathbf{R}_1\mathbf{R}_2\boldsymbol{\rho}_2$, so the velocity of it can be written as

$$v_2 = \dot{\mathbf{R}}_1 \mathbf{q}_1 + \dot{\mathbf{R}}_1 \mathbf{R}_2 \boldsymbol{\rho}_2 + \mathbf{R}_1 \dot{\mathbf{R}}_2 \boldsymbol{\rho}_2 = \omega_1 \mathbf{R}_1 \mathbf{S}_1 \mathbf{q}_1 + (\omega_1 + \omega_2) \mathbf{R}_1 \mathbf{R}_2 \mathbf{S}_1 \boldsymbol{\rho}_2 \tag{3.3}$$

So the kinetic and potential energy of BC is

$$\begin{aligned}\mathbf{T}_2 &= \frac{1}{2}(J_2 + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\boldsymbol{\rho}_2)(\omega_1+\omega_2)^2 + \frac{1}{2}m_2\omega_1^2\mathbf{q}_1^{\mathrm{T}}\mathbf{q}_1 + m_2(\omega_1+\omega_2)\omega_1\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1 \\ V_2 &= -m_2 g\mathbf{e}_2^{\mathrm{T}}(\mathbf{R}_1\mathbf{q}_2 + \mathbf{R}_1\mathbf{R}_2\boldsymbol{\rho}_2)\end{aligned} \tag{3.4}$$

The third term of kinetic energy can be reduced by $\mathbf{S}_1^{\mathrm{T}}\mathbf{R}\mathbf{S}_1 = \mathbf{R}$. According to equations (3.2) and (3.4), the Lagrange function of the system is

$$\begin{aligned}L &= \frac{1}{2}(J_2 + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\boldsymbol{\rho}_2)(\omega_1+\omega_2)^2 + \frac{1}{2}m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{q}_1\omega_1^2 + \frac{1}{2}(J_1 + m_1\boldsymbol{\rho}_1^{\mathrm{T}}\boldsymbol{\rho}_1)\omega_1^2 \\ &\quad + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{q}_1(\omega_1+\omega_2)\omega_1 + m_2 g\mathbf{e}_2^{\mathrm{T}}(\mathbf{R}_1\mathbf{q}_1 + \mathbf{R}_1\mathbf{R}_2\boldsymbol{\rho}_2) + m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\boldsymbol{\rho}_1\end{aligned} \tag{3.5}$$

Make variation to the angular velocity of the Lagrange function as

$$\mathbf{D}_{\omega_1}L_C\delta\mathbf{S}(\omega_1) = J_A\omega_2 + J_B\omega_1 \tag{3.6}$$

In (3.6), $J_A = J_1 + m_1\|\boldsymbol{\rho}_1\|^2 + m_2\|\mathbf{q}_1\|^2 + J_B + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{q}_1$, $J_B = J_2 + m_2\|\boldsymbol{\rho}_2\|^2 + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{q}_1$. Differentiate then equation (3.6), which gives

$$\begin{aligned}\frac{d}{dt}(\mathbf{D}_{\omega_1}L_C) &= J_A\dot{\omega}_2 + J_B\dot{\omega}_1 + \dot{J}_A\omega_2 + \dot{J}_B\omega_1 = J_A\dot{\omega}_2 + J_B\dot{\omega}_1 \\ &\quad - m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_2^2 - 2m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_1\omega_2\end{aligned} \tag{3.7}$$

In (3.7), $\dot{J}_A = 2\dot{J}_B = -2m_2\omega_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_2^{\mathrm{T}}\mathbf{q}_1$. Make variation to ω_2

$$\mathbf{D}_{\omega_2}L_C\cdot\delta\mathbf{S}(\omega_2) = J_C\omega_1 + J_D\omega_2 \tag{3.8}$$

In (3.8), $J_C = J_D + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{q}_1$, $J_D = J_2 + m_2\|\boldsymbol{\rho}_2\|^2$. Differentiate (3.8)

$$\frac{d}{dt}(\mathbf{D}_{\omega_2}L_C) = J_C\dot{\omega}_1 + J_D\dot{\omega}_2 - (m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1)\omega_1\omega_2 \tag{3.9}$$

Take variation to $\mathbf{R}_1$, $\mathbf{R}_2$, then find the tangent vector of them. The results is

$$\begin{aligned}\mathbf{T}_e^*L_{\mathbf{R}_1}\cdot\mathbf{D}_{\mathbf{R}_1}L &= m_2 g\mathbf{e}_2^{\mathrm{T}}(\mathbf{R}_1\mathbf{S}_1\mathbf{q}_1 + \mathbf{R}_1\mathbf{S}_1\mathbf{R}_2\boldsymbol{\rho}_2) + m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{S}_1\boldsymbol{\rho}_1 \\ \mathbf{T}_e^*L_{\mathbf{R}_2}\cdot\mathbf{D}_{\mathbf{R}_2}L &= m_2 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{R}_2\mathbf{S}_1\boldsymbol{\rho}_2 - m_2(\omega_1+\omega_2)\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_2\boldsymbol{\rho}_2\end{aligned} \tag{3.10}$$

From (3.7), (3.9) and (3.10), the Lagrange equation is

$$\begin{aligned}&J_A\dot{\omega}_2 + J_B\dot{\omega}_1 + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_2^2 - m_2 g\mathbf{e}_2^{\mathrm{T}}(\mathbf{R}_1\mathbf{S}_1\mathbf{q}_1 + \mathbf{R}_1\mathbf{S}_1\mathbf{R}_2\boldsymbol{\rho}_2) + 2m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_1\omega_2 \\ &\qquad - m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{S}_1\boldsymbol{\rho}_1 = 0 \\ &J_C\dot{\omega}_1 + J_D\dot{\omega}_2 + (m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1)\omega_1\omega_2 - m_2 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{R}_2\mathbf{S}_1\boldsymbol{\rho}_2 - m_2(\omega_1+\omega_2)\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_2\boldsymbol{\rho}_2 = 0\end{aligned} \tag{3.11}$$

Simplify it to

$$\begin{aligned}\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\end{bmatrix} &= -\begin{bmatrix}J_B & J_A\\ J_C & J_D\end{bmatrix}^{-1}(\mathbf{Q}_A + \mathbf{Q}_B) \\ \mathbf{Q}_A &= \begin{bmatrix}m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_2^2 + 2m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_1\omega_2 \\ m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_1\omega_2\end{bmatrix} \\ \mathbf{Q}_B &= \begin{bmatrix}-m_2 g\mathbf{e}_2^{\mathrm{T}}(\mathbf{R}_1\mathbf{S}_1\mathbf{q}_1 + \mathbf{R}_1\mathbf{S}_1\mathbf{R}_2\boldsymbol{\rho}_2) - m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{S}_1\boldsymbol{\rho}_1) \\ -m_2 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{R}_2\mathbf{S}_1\boldsymbol{\rho}_2 - m_2(\omega_1+\omega_2)\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_2\boldsymbol{\rho}_2)\end{bmatrix}\end{aligned} \tag{3.12}$$

Take the Legendre transformation to (3.12), supposing that $\Pi_1 = J_A\omega_2 + J_B\omega_1$, $\Pi_2 = J_C\omega_1 + J_D\omega_2$. The dynamical equation of the system are

$$\begin{aligned}&\dot{\Pi}_1 - m_2 g\mathbf{e}_2^{\mathrm{T}}(\mathbf{R}_1\mathbf{S}_1\mathbf{q}_1 + \mathbf{R}_1\mathbf{R}_2\mathbf{S}_1\boldsymbol{\rho}_2) - m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{S}_1\boldsymbol{\rho}_1 = 0 \\ &\dot{\Pi}_2 - m_2 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{R}_2\mathbf{S}_1\boldsymbol{\rho}_2 - m_2(\omega_1+\omega_2)\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_2\boldsymbol{\rho}_2 = 0\end{aligned} \tag{3.13}$$

4. Dynamic modeling with absolute motion

In this part, the absolute motion parameters are used to dynamical modeling of the double pendulum. The absolute attitude matrices of AB, BC are $\mathbf{R}_A$, $\mathbf{R}_B$, respectively. ω_A, ω_B are absolute angular velocities. The kinetic and potential energy of the first pendulum is

$$T_1 = \frac{1}{2}(J_1 + m_1\boldsymbol{\rho}_1^{\mathrm{T}}\boldsymbol{\rho}_1)\omega_A^2 V_1 = -m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\boldsymbol{\rho}_1 \tag{4.1}$$

According to the attitude transformation, the position of mass center of BC is $\mathbf{R}_A\mathbf{q}_1+\mathbf{R}_B\boldsymbol{\rho}_2$, so the velocity of mass center of BC is

$$v_2 = \dot{\mathbf{R}}_A\mathbf{q}_1 + \dot{\mathbf{R}}_B\boldsymbol{\rho}_2 = \omega_A\mathbf{R}_A\mathbf{S}_1\mathbf{q}_1 + \omega_B\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2 \tag{4.2}$$

Square (4.2) to obtain

$$\|v_2\|^2 = \omega_A^2\mathbf{q}_1^{\mathrm{T}}\mathbf{q}_1 + \omega_B^2\boldsymbol{\rho}_2^{\mathrm{T}}\boldsymbol{\rho}_2 + 2\omega_A\omega_B\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2 \tag{4.3}$$

So the kinetic and potential energy of BC is

$$\begin{aligned} T_2 &= \frac{1}{2}J_2\omega_B^2 + \frac{1}{2}m_2\|v_2\|^2 = \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}m_2\omega_A^2\mathbf{q}_1^{\mathrm{T}}\mathbf{q}_1 + \frac{1}{2}m_2\omega_B^2\boldsymbol{\rho}_2^{\mathrm{T}}\boldsymbol{\rho}_2 \\ &\quad + m_2\omega_A\omega_B\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2 \\ V_2 &= -m_2 g\mathbf{e}_2^{\mathrm{T}}(\mathbf{R}_A\mathbf{q}_2 + \mathbf{R}_B\boldsymbol{\rho}_2) \end{aligned} \tag{4.4}$$

The Lagrange function of the system is

$$\begin{aligned} L &= \frac{1}{2}(J_1 + m_1\boldsymbol{\rho}_1^{\mathrm{T}}\boldsymbol{\rho}_1 + m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{q}_1)\omega_A^2 + \frac{1}{2}(J_2 + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\boldsymbol{\rho}_2)\omega_B^2 + m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\boldsymbol{\rho}_1 \\ &\quad + m_2\omega_A\omega_B\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2 + m_2 g\mathbf{e}_2^{\mathrm{T}}(\mathbf{R}_A\mathbf{q}_2 + \mathbf{R}_B\boldsymbol{\rho}_2) \end{aligned} \tag{4.5}$$

Take variation to the angular velocities

$$\begin{aligned} &\mathbf{D}_{\omega_A}L_C \cdot \delta\mathbf{S}(\omega_A) = (J_1 + m_1\boldsymbol{\rho}_1^{\mathrm{T}}\boldsymbol{\rho}_1 + m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{q}_1)\omega_A + m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\omega_B \\ &\mathbf{D}_{\omega_B}L_C \cdot \delta\mathbf{S}(\omega_B) = (J_2 + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\boldsymbol{\rho}_2)\omega_B + m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\omega_A \end{aligned} \tag{4.6}$$

According to equations (4.6), the last terms in (4.6) have similar expressions. After differentiation, one obtains

$$\frac{d}{dt}(m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2) = m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2(\omega_A - \omega_B) \tag{4.7}$$

Equation (4.7) is reduced by $\mathbf{R}\mathbf{S}_1 = -\mathbf{S}_1^{\mathrm{T}}\mathbf{R}$. Take variation to the rotation matrix

$$\begin{aligned} &\mathbf{T}_e^*L_{\mathbf{R}_A} \cdot \mathbf{D}_{\mathbf{R}_A}L = -m_2\omega_A\omega_B\mathbf{q}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2 + m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\mathbf{S}_1\boldsymbol{\rho}_1 + m_2 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\mathbf{S}_1\mathbf{q}_2 \\ &\mathbf{T}_e^*L_{\mathbf{R}_B} \cdot \mathbf{D}_{\mathbf{R}_B}L = m_2\omega_A\omega_B\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\boldsymbol{\rho}_2 + m_2 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2 \end{aligned} \tag{4.8}$$

Then the dynamical equation of the system is

$$\begin{aligned} &(J_1 + m_1\boldsymbol{\rho}_1^{\mathrm{T}}\boldsymbol{\rho}_1 + m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{q}_1)\dot{\omega}_A - m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\dot{\omega}_B + m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\boldsymbol{\rho}_2\omega_B^2 \\ &\quad - m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\omega_A\omega_B + m_2\omega_A\omega_B\mathbf{q}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2 \\ &\quad - m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\mathbf{S}_1\boldsymbol{\rho}_1 - m_2 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\mathbf{S}_1\mathbf{q}_2 = 0 \\ &(J_2 + m_2\boldsymbol{\rho}_2^{\mathrm{T}}\boldsymbol{\rho}_2)\dot{\omega}_B - m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\dot{\omega}_A + m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\boldsymbol{\rho}_2\omega_A\omega_B \\ &\quad - m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\omega_A^2 - m_2\omega_A\omega_B\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\boldsymbol{\rho}_2 - m_2 g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2 = 0 \end{aligned} \tag{4.9}$$

According to the Legendre transformation, and supposing that $\mathit{\Pi}_A = \mathbf{D}_{\omega_A} L$, $\mathit{\Pi}_B = \mathbf{D}_{\omega_B} L$, the dynamical equations can be simplified as

$$\begin{aligned}
&\dot{\mathit{\Pi}}_A - m_B \omega_A \omega_B \mathbf{q}_1^{\mathrm{T}} \mathbf{R}_A^{\mathrm{T}} \mathbf{R}_B \mathbf{S}_1 \boldsymbol{\rho}_2 + m_1 g \mathbf{e}_2^{\mathrm{T}} \mathbf{R}_A \mathbf{S}_1 \boldsymbol{\rho}_1 + m_2 g \mathbf{e}_2^{\mathrm{T}} \mathbf{R}_A \mathbf{S}_1 \mathbf{q}_2 = 0 \\
&\dot{\mathit{\Pi}}_B + m_B \omega_A \omega_B \mathbf{q}_1^{\mathrm{T}} \mathbf{S}_1 \mathbf{R}_A^{\mathrm{T}} \mathbf{R}_B \boldsymbol{\rho}_2 + m_2 g \mathbf{e}_2^{\mathrm{T}} \mathbf{R}_B \mathbf{S}_1 \boldsymbol{\rho}_2 = 0
\end{aligned} \tag{4.10}$$

Comparing (3.13) and (4.10), both of them are reduced in complexity, which can may obviously reduce the complexity of solving.

5. Dynamic modeling with the attitude vector

In this Section, the dynamical equations are built with the attitude vector. Supposing the attitude of AB is $\mathbf{q}_1$, the attitude vector of BC relative to AB is $\mathbf{q}_2$, $\mathbf{q}_1$ and $\mathbf{q}_2$ are both unit vectors. Their derivatives are

$$\dot{\mathbf{q}}_1 = \omega_1 \mathbf{S}_1 \mathbf{q}_1 \qquad \dot{\mathbf{q}}_2 = \omega_2 \mathbf{S}_1 \mathbf{q}_2 \tag{5.1}$$

The absolute attitude vector of BC is

$$\mathbf{q}_B = \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_1 \mathbf{q}_2 \mathbf{e}_1 + \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_2 \mathbf{q}_2 \mathbf{e}_2 \tag{5.2}$$

In (5.2), $\mathbf{T}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{T}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Differentiating (5.2), the velocity is

$$\dot{\mathbf{q}}_B = (\omega_1 + \omega_2) \mathbf{S}_1 \mathbf{q}_B \tag{5.3}$$

Suppose that the length of AB and BC are l_1 and l_2, respectively. The mass center lies on the direction line of AB and BC, and the position of mass centers are $\alpha_1 \mathbf{q}_1 l_1$ and $\mathbf{q}_1 l_1 + \alpha_2 \mathbf{q}_B l_2$, respectively, which is very simple. If the position of mass centers in its own coordinates of AB and BC are $\boldsymbol{\rho}_1 = [x_1, y_1]$ and $\boldsymbol{\rho}_2 = [x_2, y_2]$, respectively, then the positions of the mass centers of AB and BC are $\mathbf{T}_{\rho_1} \mathbf{q}_1$ and $\mathbf{q}_1 l_1 + \mathbf{T}_{\rho_1} \mathbf{q}_B$, respectively. The expressions of T_{ρ_1}, T_{ρ_2} are

$$\mathbf{T}_{\rho_1} = \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} \qquad \mathbf{T}_{\rho_2} = \begin{bmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{bmatrix} \tag{5.4}$$

Then the velocity of two mass centers are

$$\begin{aligned}
\mathbf{v}_{c1} &= \mathbf{T}_{\rho_1} \dot{\mathbf{q}}_1 = \omega_1 \mathbf{T}_{\rho_1} \mathbf{S}_1 \mathbf{q}_1 \\
\mathbf{v}_{c2} &= \dot{\mathbf{q}}_1 l_1 + \mathbf{T}_{\rho_1} \dot{\mathbf{q}}_B = \omega_1 \mathbf{S}_1 \mathbf{q}_1 l_1 + \omega_B \mathbf{T}_{\rho_1} \mathbf{S}_1 \mathbf{q}_B
\end{aligned} \tag{5.5}$$

The Lagrange function of the system is

$$\begin{aligned}
L &= \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} m_1 \|\mathbf{v}_{c1}\|^2 + \frac{1}{2} J_2 \omega_B^2 + \frac{1}{2} m_2 \|\mathbf{v}_{c2} t\|^2 + m_1 g \mathbf{e}_2^{\mathrm{T}} \mathbf{T}_{\rho_1} \mathbf{q}_1 + m_2 g \mathbf{e}_2^{\mathrm{T}} (\mathbf{q}_1 l_1 + \mathbf{T}_{\rho_2} \mathbf{q}_B) \\
&= \frac{1}{2} (J_1 + m_1 \|\boldsymbol{\rho}_1\|^2 + m_2 l_1^2) \omega_1^2 + \frac{1}{2} (J_2 + m_2 \|\boldsymbol{\rho}_1\|^2) \omega_B^2 \\
&\quad + m_2 l_1 \mathbf{q}_B^{\mathrm{T}} \mathbf{T}_{\rho_2}^{\mathrm{T}} \mathbf{q}_1 \omega_B \omega_1 + m_1 g \mathbf{e}_2^{\mathrm{T}} \mathbf{T}_{\rho_1} \mathbf{q}_1 + m_2 g \mathbf{e}_2^{\mathrm{T}} (\mathbf{q}_1 l_1 + \mathbf{T}_{\rho_2} \mathbf{q}_B)
\end{aligned} \tag{5.6}$$

Take variation to ω_1, ω_2

$$\mathbf{D}_{\omega_1} L \cdot \delta \mathbf{S}(\omega_1) = J_{Q1} \omega_1 + J_{QB} \omega_2 \qquad \mathbf{D}_{\omega_2} L \cdot \delta \mathbf{S}(\omega_2) = J_{QB} \omega_1 + J_{QA} \omega_2 \tag{5.7}$$

In (5.7), $J_{Q1} = J_1 + m_2 l_1^2 + m_1\|\boldsymbol{\rho}_1\|^2 + J_{QA} + 2m_2 l_1 \mathbf{q}_B^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}} \mathbf{q}_1$. The derivatives of J_{QB} and J_{Q1} are

$$\dot{J}_{QB} = m_2 l_1(\omega_1 \mathbf{q}_B^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}} \mathbf{S}_1 \mathbf{q}_1 - \omega_B \mathbf{q}_B^{\mathrm{T}} \mathbf{S}_1 \mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}} \mathbf{q}_1) \qquad \dot{J}_{Q1} = 2\dot{J}_{QB} \tag{5.8}$$

Differentiate the variation of ω

$$\begin{aligned}
&\frac{d}{dt}\mathbf{D}_{\omega_1} L \cdot \delta\mathbf{S}(\omega_1) = J_{Q1}\dot{\omega}_1 + J_{QB}\dot{\omega}_2 + \dot{J}_{Q1}\omega_1 + \dot{J}_{QB}\omega_2 = J_{Q1}\dot{\omega}_1 + J_{QB}\dot{\omega}_2 + \dot{J}_{QB}(2\omega_1 + \omega_2) \\
&\frac{d}{dt}\mathbf{D}_{\omega_2} L \cdot \delta\mathbf{S}(\omega_2) = J_{QB}\dot{\omega}_1 + J_{QA}\dot{\omega}_2 + \dot{J}_{QB}\omega_1
\end{aligned} \tag{5.9}$$

Find variation of the attitude parameters in the Lagrange function. Take variation to $\mathbf{q}_B$

$$\delta\mathbf{q}_B = \delta\mathbf{q}_1^{\mathrm{T}} \mathbf{T}_1 \mathbf{q}_2 \mathbf{e}_1 + \delta\mathbf{q}_1^{\mathrm{T}} \mathbf{T}_2 \mathbf{q}_2 \mathbf{e}_2 + \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_1 \delta\mathbf{q}_2 \mathbf{e}_1 + \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_2 \delta\mathbf{q}_2 \mathbf{e}_2 \tag{5.10}$$

According to the transformation relation of the middle matrix, (5.10) is changed to

$$\begin{aligned}
\delta\mathbf{q}_B &= \delta\mathbf{q}_1^{\mathrm{T}} \mathbf{T}_1 \mathbf{q}_2 \mathbf{e}_1 + \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_1 \delta\mathbf{q}_2 \mathbf{e}_1 + \delta\mathbf{q}_1^{\mathrm{T}} \mathbf{T}_2 \mathbf{q}_2 \mathbf{e}_2 + \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_2 \delta\mathbf{q}_2 \mathbf{e}_2 \\
&= [(\mathbf{q}_1^{\mathrm{T}} \mathbf{T}_1 \mathbf{q}_2)\mathbf{e}_2 - (\mathbf{q}_1^{\mathrm{T}} \mathbf{T}_2 \mathbf{q}_2)\mathbf{e}_1](\eta_1 + \eta_2)
\end{aligned} \tag{5.11}$$

Take variation to the Lagrange function

$$\begin{aligned}
\delta_q L = {}& m_2(l_1 \omega_B \omega_1 \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2} + g\mathbf{e}_2^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2})\delta\mathbf{q}_B + (m_2 l_1 \omega_B \omega_1 \mathbf{q}_B^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}} + m_1 g \mathbf{e}_2^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_1} \\
&+ m_2 g l_1 \mathbf{e}_2^{\mathrm{T}})\delta\mathbf{q}_1
\end{aligned} \tag{5.12}$$

Take (5.11) into (5.12), the variation of L is

$$\delta_q L = (w_1 w_2 + w_3 \mathbf{S}_1 \mathbf{q}_1)\eta_1 + w_1 w_2 \eta_2 \tag{5.13}$$

The parameters in (5.13) are as follows

$$\begin{aligned}
w_1 &= m_2 l_1 \omega_B \omega_1 \mathbf{q}_1^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2} + m_2 g \mathbf{e}_2^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2} \\
w_2 &= (\mathbf{q}_1^{\mathrm{T}} \mathbf{T}_1 \mathbf{q}_2)\mathbf{e}_2 - (\mathbf{q}_1^{\mathrm{T}} \mathbf{T}_2 \mathbf{q}_2)\mathbf{e}_1 \\
w_3 &= m_2 l_1 \omega_B \omega_1 \mathbf{q}_B^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}} + m_1 g \mathbf{e}_2^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_1} + m_2 g l_1 \mathbf{e}_2^{\mathrm{T}}
\end{aligned}$$

Then the tangent projections are

$$\mathbf{T}_e^* L_{\mathbf{q}_1} \cdot \mathbf{D}_{\mathbf{q}_1} L = w_1 w_2 + w_3 \mathbf{S}_1 \mathbf{q}_1 \qquad \mathbf{T}_e^* L_{\mathbf{q}_2} \cdot \mathbf{D}_{\mathbf{q}_2} L = w_1 w_2 \tag{5.14}$$

The dynamical equation of the system is as

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \dot{\mathbf{J}}_{QB}\mathbf{N}\boldsymbol{\omega} - \mathbf{W} = \mathbf{0} \tag{5.15}$$

The parameters in (5.17) are as follows

$$\mathbf{J} = \begin{bmatrix} J_{Q1} & J_{QB} \\ J_{QB} & J_{QA} \end{bmatrix} \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} w_1 w_2 + w_3 \mathbf{S}_1 \mathbf{q}_1 \\ w_1 w_2 \end{bmatrix}$$

Take variation to absolute angular velocities in the Lagrange function, as

$$\begin{aligned}
&\mathbf{D}_{\omega_1} L \cdot \delta\mathbf{S}(\omega_1) = (J_1 + m_1\|\boldsymbol{\rho}_1\|^2 + m_2 l_1^2)\omega_1 + m_2 l_1 \mathbf{q}_B^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}} \mathbf{q}_1 \omega_B \\
&\mathbf{D}_{\omega_B} L \cdot \delta\mathbf{S}(\omega_B) = (J_2 + m_2\|\boldsymbol{\rho}_1\|^2)\omega_B + m_2 l_1 \mathbf{q}_B^{\mathrm{T}} \mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}} \mathbf{q}_1 \omega_1
\end{aligned} \tag{5.16}$$

Find derivative of equation (5.16), as

$$\begin{aligned}&\frac{d}{dt}\mathbf{D}_{\omega_B}L\cdot\delta\mathbf{S}(\omega_B)=(J_2+m_2\|\boldsymbol{\rho}_1\|^2)\dot{\omega}_B+m_2l_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{q}_1\dot{\omega}_1\\&\qquad+m_2l_1(\omega_1^2\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1-\omega_B\omega_1\mathbf{q}_B^{\mathrm{T}}\mathbf{S}_1\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{q}_1)\end{aligned}\tag{5.17}$$

The tangent projection are

$$\begin{aligned}&\mathbf{T}_e^*L_{\mathbf{q}_1}\cdot\mathbf{D}_{\mathbf{q}_1}L=(m_2l_1\omega_B\omega_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}+m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_1}+m_2gl_1\mathbf{e}_2^{\mathrm{T}})\mathbf{S}_1\mathbf{q}_1\\&\mathbf{T}_e^*L_{\mathbf{q}_B}\cdot\mathbf{D}_{\mathbf{q}_B}L=(m_2l_1\omega_B\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}+m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2})\delta\mathbf{q}_B\\&\qquad=(m_2l_1\omega_B\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}+m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2})\mathbf{S}_1\mathbf{q}_B\end{aligned}\tag{5.18}$$

At last, the Lagrange equation with the absolute attitude expression is

$$\begin{aligned}&(J_1+m_1\|\boldsymbol{\rho}_1\|^2+m_2l_1^2)\dot{\omega}_1+m_2l_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{q}_1\dot{\omega}_B-m_2l_1\omega_B^2\mathbf{S}_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{q}_1\\&\qquad+m_2l_1\omega_1\omega_B\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1-(m_2l_1\omega_B\omega_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}-m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_1}-m_2gl_1\mathbf{e}_2^{\mathrm{T}})\mathbf{S}_1\mathbf{q}_1=0\\&(J_2+m_2\|\boldsymbol{\rho}_1\|^2)\dot{\omega}_B+m_2l_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{q}_1\dot{\omega}_1+m_2l_1(\omega_1^2\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1-\omega_B\omega_1\mathbf{q}_B^{\mathrm{T}}\mathbf{S}_1\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{q}_1)\\&\qquad-(m_2l_1\omega_B\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}-m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2})\mathbf{S}_1\mathbf{q}_B=0\end{aligned}\tag{5.19}$$

According to the Legendre transformation, the dynamical equation with the relative attitude satisfies the relation as $\mathbf{D}_{\omega_1}L\cdot\delta\mathbf{S}(\omega_1)=\Pi_1$, $\mathbf{D}_{\omega_2}L\cdot\delta\mathbf{S}(\omega_2)=\Pi_2$. So, the Hamilton equation with the relative attitudes is

$$\begin{aligned}&\dot{\Pi}_1+m_2l_1\omega_B\omega_1(\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1^{\mathrm{T}}-\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2^{\mathrm{T}})\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{q}_1-m_2l_1\omega_B\omega_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1-m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_1}\mathbf{S}_1\mathbf{q}_1\\&\qquad-m_2g(\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1-\mathbf{e}_1^{\mathrm{T}}\mathbf{q}_1l_1-\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2)=0\\&\dot{\Pi}_2-m_2l_1\omega_B\omega_1(\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2^{\mathrm{T}}-\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1^{\mathrm{T}})\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}\mathbf{q}_1+m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}(\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2-\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1)=0\end{aligned}\tag{5.20}$$

In a similar way, the Hamilton equation with the absolute parameters is

$$\begin{aligned}&\dot{\Pi}_1-(m_2l_1\omega_B\omega_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}^{\mathrm{T}}-m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_1}-m_2gl_1\mathbf{e}_2^{\mathrm{T}})\mathbf{S}_1\mathbf{q}_1=0\\&\dot{\Pi}_B-(m_2l_1\omega_B\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2}-m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\boldsymbol{\rho}_2})\mathbf{S}_1\mathbf{q}_B=0\end{aligned}\tag{5.21}$$

Based on the Lie group and attitude vector, eight types of dynamical equations are derived from the relative and absolute motion and Lagrange and Hamilton theory. They are named in the following format X-Y-Z. X represents the modeling method as the Lie group or attitude vector, Y represents the type of motion as relative or absolute, and Z represents the type as Lagrange or Hamilton. So the names of these eight equations are: *L-R-L*; *L-R-H*; *L-A-L*; *L-A-H*; *A-R-L*; *A-R-H*; *A-A-L*; *A-A-H*. From the above results, it occure that the Hamilton equations are much simpler than the Lagrange ones, which can enhance the computation efficiency. According to the comparison of the Lagrange equation with the absolute and relative motion, the complexity of the equation is not obviously reduced. For the expression, the attitude vector gives a simpler form than the Lie group. So from the point of view of complexity, the attitude vector is better than the Lie group, and the Hamilton equation is better than Lagrange one. So the dynamical equation based on the attitude vector, Hamilton theory, yields the most simple expression.

6. Numerical computations

The eight dynamical equations need to be arranged and completed with some conditions before numerical computations. From the point of view of ordinary differential equation computations, the equations should be given some conditions which satisfy the number of unknown parameters equal to the dimensions. For the L-A-L equation, there are many solution methods according to the parameter selection. The first and the most visual is to select ω_1, ω_2, θ_1, θ_2 as the parameters. It can be obtained by an exponential map. It is the most custom one. Here the geometry expression is chosen to avoid triangle computations. The kinematic expressions based on the attitude matrix are

$$\dot{\mathbf{R}}_1 = \mathbf{R}_1\mathbf{S}_1\omega_1 \qquad \dot{\mathbf{R}}_2 = \mathbf{R}_2\mathbf{S}_1\omega_2 \tag{6.1}$$

Equation (6.1) can not be used in the dynamical equation directly. Multiplying $\mathbf{e}_2^{\mathrm{T}}$ on both sides of (6.1), the vector equations can be written as

$$\dot{\mathbf{R}}_1^{\mathrm{T}}\mathbf{e}_2 = -\mathbf{S}_1\mathbf{R}_1^{\mathrm{T}}\mathbf{e}_2\omega_1 \qquad \dot{\mathbf{R}}_2^{\mathrm{T}}\mathbf{e}_2 = -\mathbf{S}_1\mathbf{R}_2^{\mathrm{T}}\mathbf{e}_2\omega_2 \tag{6.2}$$

Equation (6.2) changes to (6.3) by $\mathit{\Gamma}_1 = \mathbf{R}_1^{\mathrm{T}}\mathbf{e}_2$, $\mathit{\Gamma}_2 = \mathbf{R}_2^{\mathrm{T}}\mathbf{e}_2$

$$\dot{\mathit{\Gamma}}_1 = -\mathbf{S}_1\mathit{\Gamma}_1\omega_1 \qquad \dot{\mathit{\Gamma}}_2 = -\mathbf{S}_1\mathit{\Gamma}_2\omega_2 \tag{6.3}$$

In the matrix form

$$\begin{bmatrix}\dot{\mathit{\Gamma}}_1\\ \dot{\mathit{\Gamma}}_2\end{bmatrix} = -\begin{bmatrix}\mathbf{S}_1\mathit{\Gamma}_1 & \mathbf{O}_{2\times1}\\ \mathbf{O}_{2\times1} & \mathbf{S}_1\mathit{\Gamma}_2\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \tag{6.4}$$

So the dynamical equations are transformed to be the ODEs with dimension of 6

$$\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\end{bmatrix} = \begin{bmatrix}J_B & J_A\\ J_C & J_D\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{T}_1\\ \mathbf{T}_2\end{bmatrix} \qquad \begin{bmatrix}\dot{\mathit{\Gamma}}_1\\ \dot{\mathit{\Gamma}}_2\end{bmatrix} = -\begin{bmatrix}\mathbf{S}_1\mathit{\Gamma}_1 & \mathbf{O}_{2\times1}\\ \mathbf{O}_{2\times1} & \mathbf{S}_1\mathit{\Gamma}_2\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \tag{6.5}$$

The terms in (6.5) are as follows

$$\begin{aligned}\mathbf{T}_1 &= m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_2^2 + 2m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_1\omega_2 + m_2g\mathbf{e}_2^{\mathrm{T}}(\mathbf{R}_1\mathbf{S}_1\mathbf{q}_2\\ &\quad + \mathbf{R}_1\mathbf{R}_2\mathbf{S}_1\boldsymbol{\rho}_2) + m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{S}_1\boldsymbol{\rho}_1\\ \mathbf{T}_2 &= m_2\boldsymbol{\rho}_2^{\mathrm{T}}\mathbf{R}_2^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\omega_2\omega_1 + m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{R}_2\mathbf{S}_1\boldsymbol{\rho}_2 + m_2(\omega_1+\omega_2)\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_2\boldsymbol{\rho}_2\end{aligned}$$

The inverse matrix of (6.5) can be found directly to decrease the amount computations during numerical solution

$$\begin{bmatrix}J_B & J_A\\ J_C & J_D\end{bmatrix}^{-1} = \frac{1}{J_BJ_D - J_AJ_C}\begin{bmatrix}J_D & -J_A\\ -J_C & J_D\end{bmatrix} \tag{6.6}$$

In the *L-A-H* equation, $\mathit{\Pi}_1$, $\mathit{\Pi}_2$, $\mathbf{R}_1$, $\mathbf{R}_2$ are unknown parameters, so the angular velocities need to be eliminated. According to the relation between $\mathit{\Pi}_1$, $\mathit{\Pi}_2$ and ω_1, ω_2, the angular velocities are

$$\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} = \begin{bmatrix}J_B & J_A\\ J_C & J_D\end{bmatrix}^{-1}\begin{bmatrix}\mathit{\Pi}_1\\ \mathit{\Pi}_2\end{bmatrix} \tag{6.7}$$

Substitute (6.7) into (6.5), the dynamical equation is then

$$\begin{bmatrix}\dot{\mathit{\Gamma}}_1\\ \dot{\mathit{\Gamma}}_2\end{bmatrix} = -\begin{bmatrix}\mathbf{S}_1\mathit{\Gamma}_1 & \mathbf{O}_{2\times1}\\ \mathbf{O}_{2\times1} & \mathbf{S}_1\mathit{\Gamma}_2\end{bmatrix}\begin{bmatrix}J_B & J_A\\ J_C & J_D\end{bmatrix}^{-1}\begin{bmatrix}\mathit{\Pi}_1\\ \mathit{\Pi}_2\end{bmatrix} \tag{6.8}$$

Combine (6.8) with the L-A-H equation, the dynamical equations with dimension of 6 will be

$$\begin{aligned}\begin{bmatrix}\dot{\Pi}_1\\\dot{\Pi}_2\end{bmatrix}&=\begin{bmatrix}m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{R}_2\mathbf{S}_1\boldsymbol{\rho}_2+m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{S}_1\mathbf{q}_2+m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{S}_1\boldsymbol{\rho}_1\\ m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_1\mathbf{R}_2\mathbf{S}_1\boldsymbol{\rho}_2+m_2(\omega_1+\omega_2)\omega_1\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_2\boldsymbol{\rho}_2\end{bmatrix}\\ \begin{bmatrix}\dot{\Gamma}_1\\\dot{\Gamma}_2\end{bmatrix}&=-\begin{bmatrix}\mathbf{S}_1\Gamma_1 & \mathbf{O}_{2\times1}\\\mathbf{O}_{2\times1} & \mathbf{S}_1\Gamma_2\end{bmatrix}\begin{bmatrix}J_B & J_A\\ J_C & J_D\end{bmatrix}^{-1}\begin{bmatrix}\Pi_1\\\Pi_2\end{bmatrix}\end{aligned}\tag{6.9}$$

In (6.9), $\mathbf{R}_1=[-\mathbf{S}_1\Gamma_1,\Gamma_1]$, $\mathbf{R}_2=[-\mathbf{S}_1\Gamma_2,\Gamma_2]$. Similarly, L-A-L can also be written as an explicit formulation

$$\dot{\boldsymbol{\omega}}=\mathbf{J}_K^{-1}\mathbf{K}\qquad\qquad \dot{\mathbf{\Gamma}}=\mathbf{S}_\Gamma\boldsymbol{\omega}\tag{6.10}$$

In (6.10)

$$\begin{aligned}&\boldsymbol{\omega}=[\omega_A,\omega_B]\qquad\qquad \mathbf{\Gamma}=[\Gamma_A,\Gamma_B]\qquad\qquad \mathbf{S}_\Gamma=\begin{bmatrix}\mathbf{S}_1\Gamma_1 & \mathbf{O}_{2\times1}\\\mathbf{O}_{2\times1} & \mathbf{S}_1\Gamma_2\end{bmatrix}\\ &\mathbf{J}_K=\begin{bmatrix}J_1+m_1\boldsymbol{\rho}_1^{\mathrm{T}}\boldsymbol{\rho}_1+m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{q}_1 & -m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\\ -m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2 & J_2+m_2\boldsymbol{\rho}_2^{\mathrm{T}}\boldsymbol{\rho}_2\end{bmatrix}\\ &\mathbf{K}=\begin{bmatrix}m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\mathbf{S}_1\boldsymbol{\rho}_1+m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\mathbf{S}_1\mathbf{q}_2-m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\boldsymbol{\rho}_2\omega_B^2\\ m_2\mathbf{q}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\omega_A^2+m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\end{bmatrix}\end{aligned}$$

Similarly, accoding to $\mathbf{J}_K^{-1}\mathbf{\Pi}=\boldsymbol{\omega}$, the Hamilton equation with the absolute matrix is

$$\begin{aligned}&\dot{\mathbf{Pi}}=-\mathbf{K}_A\qquad\qquad \dot{\mathbf{\Gamma}}=-\mathbf{S}_\Gamma\mathbf{J}_K^{-1}\mathbf{\Pi}\\ &\mathbf{K}_A=\begin{bmatrix}-m_B\omega_A\omega_B\mathbf{q}_1^{\mathrm{T}}\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2+m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\mathbf{S}_1\boldsymbol{\rho}_1+m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_A\mathbf{S}_1\mathbf{q}_2\\ m_B\omega_A\omega_B\mathbf{q}_1^{\mathrm{T}}\mathbf{S}_1\mathbf{R}_A^{\mathrm{T}}\mathbf{R}_B\boldsymbol{\rho}_2+m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{R}_B\mathbf{S}_1\boldsymbol{\rho}_2\end{bmatrix}\end{aligned}\tag{6.11}$$

Differently to the Lie group expression, the dynamics model based on the attitude vector can be calculated directly from the vector. So the kinematic model can be directly obtained by differentiating the attitude vector. So the ODE of the A-R-L equation are

$$\mathbf{J}_Q\dot{\boldsymbol{\omega}}+\mathbf{W}=\mathbf{0}\qquad\qquad \dot{\mathbf{q}}=\mathbf{S}_W\boldsymbol{\omega}\tag{6.12}$$

The terms in (6.12) are as follows

$$\begin{aligned}&\mathbf{S}_W=\begin{bmatrix}\mathbf{S}_1\mathbf{q}_1 & \mathbf{O}_{2\times1}\\\mathbf{O}_{2\times1} & \mathbf{S}_1\mathbf{q}_2\end{bmatrix}\qquad\qquad \mathbf{J}_Q=\begin{bmatrix}J_{Q1} & J_{QB}\\ J_{QB} & J_{QA}\end{bmatrix}\qquad\qquad \mathbf{W}_Q=\begin{bmatrix}W_1\\W_2\end{bmatrix}\\ &W_1=m_2l_1(2\omega_1+\omega_2)(\omega_1-\omega_B)\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1+m_2l_1\omega_B\omega_1[\mathbf{q}_1^{\mathrm{T}}(\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1^{\mathrm{T}}-\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2^{\mathrm{T}})\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{q}_1\\ &\qquad-\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1]-m_2g(\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\rho_2}\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1-\mathbf{e}_1^{\mathrm{T}}\mathbf{q}_1l_1-\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\rho_2}\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2)-m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\rho_1}\mathbf{S}_1\mathbf{q}_1\\ &W_2=m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\rho_2}(\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2-\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1)-m_2l_1\omega_B\omega_1(\mathbf{q}_B^{\mathrm{T}}\mathbf{S}_1+\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2^{\mathrm{T}}\\ &\qquad-\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1^{\mathrm{T}})\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{q}_1+m_2l_1\omega_1^2\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\end{aligned}$$

The A-R-H dynamical equation can be transformed to the following type

$$\dot{\mathbf{\Pi}}=\mathbf{U}\qquad\qquad \dot{\mathbf{q}}=\mathbf{S}_W\mathbf{J}_Q^{-1}\mathbf{Pi}\tag{6.13}$$

The terms in (6.13) are as follows

$$\begin{aligned}&\mathbf{U}=[U_1,U_2]\qquad\qquad \boldsymbol{\omega}=\mathbf{J}_Q^{-1}\mathbf{\Pi}\\ &U_1=m_2l_1\omega_B\omega_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1-m_1g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\rho_1}\mathbf{S}_1\mathbf{q}_1+m_2g(\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\rho_2}\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1-\mathbf{e}_1^{\mathrm{T}}\mathbf{q}_1l_1\\ &\qquad-\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\rho_2}\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2)-m_2l_1\omega_B\omega_1(\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1^{\mathrm{T}}-\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2^{\mathrm{T}})\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{q}_1\\ &U_2=m_2l_1\omega_B\omega_1(\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2^{\mathrm{T}}-\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1^{\mathrm{T}})\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{q}_1-m_2g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\rho_2}(\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_1\mathbf{q}_2\mathbf{e}_2-\mathbf{q}_1^{\mathrm{T}}\mathbf{T}_2\mathbf{q}_2\mathbf{e}_1)\end{aligned}$$

The *A-A-L* dynamical equation is

$$\mathbf{J}_Z\dot{\boldsymbol{\omega}} + \dot{\mathbf{J}}_Z\boldsymbol{\omega} - \mathbf{F}_Z = \mathbf{0} \qquad \dot{\mathbf{q}} = \mathbf{S}_Z\boldsymbol{\omega} \tag{6.14}$$

The following terms are

$$\mathbf{F}_Z = [F_{Z1}, F_{Z2}] \qquad \mathbf{S}_Z = \begin{bmatrix} \mathbf{S}_1\mathbf{q}_1 & \mathbf{O}_{2\times1} \\ \mathbf{O}_{2\times1} & \mathbf{S}_1\mathbf{q}_B \end{bmatrix}$$

$$\mathbf{J}_Z = \begin{bmatrix} J_1 + m_1\|\boldsymbol{\rho}_1\|^2 + m_2 l_1^2 & m_2 l_1 \mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{q}_1 \\ m_2 l_1 \mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{q}_1 & J_2 + m_2\|\boldsymbol{\rho}_1\|^2 \end{bmatrix}$$

$$\dot{\mathbf{J}}_Z = m_2 l_1(\dot{\mathbf{q}}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{q}_1 + \mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\dot{\mathbf{q}}_1)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = m_2 l_1(\omega_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1 - \mathbf{q}_B^{\mathrm{T}}\mathbf{S}_1\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{q}_1\omega_B)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= m_2 l_1(\omega_1 - \omega_B)\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}}\mathbf{S}_1\mathbf{q}_1\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

At last, the *A-A-H* dynamical equation is

$$\dot{\boldsymbol{\Pi}} = \mathbf{X} \qquad \dot{\mathbf{q}} = \mathbf{S}_Z\mathbf{J}_Z^{-1}\boldsymbol{\Pi} \tag{6.15}$$

with term as follows

$$\boldsymbol{\omega} = [\omega_1, \omega_B] = \mathbf{J}_Z^{-1}\boldsymbol{\Pi} \qquad \mathbf{X} = [X_1, X_2] \qquad \boldsymbol{\Pi} = [\Pi_1, \Pi_B]$$
$$X_1 = (m_2 l_1\omega_B\omega_1\mathbf{q}_B^{\mathrm{T}}\mathbf{T}_{\rho_2}^{\mathrm{T}} - m_1 g\mathbf{e}_2^{\mathrm{T}}\mathbf{T}_{\rho_1} - m_2 g l_1\mathbf{e}_2^{\mathrm{T}})\mathbf{S}_1\mathbf{q}_1$$
$$X_2 = m_2(l_1\omega_B\omega_1\mathbf{q}_1^{\mathrm{T}} - g\mathbf{e}_2^{\mathrm{T}})\mathbf{T}_{\rho_2}\mathbf{S}_1\mathbf{q}_B$$

According to the above analysis, the geometry dynamical equations of the double pendulum are ODEs with the dimension of six. With using the triangle function, the dimension is four. Although the dimension of the geometry model increases, it avoids the triangle transformation during differentiation, which simplifies the deriving process and programming.

In the above derivations, the dynamical models of the double pendulum are expressed by the rotation matrix $\mathbf{R}$ and the attitude vector $\mathbf{q}$. They have the same norm $\|\mathbf{R}\| = 1$, $\|\mathbf{q}\| = 1$. They represent geometrical characteristics of the dynamical system which should be conserved. From the triangle expressions of $\mathbf{R}$ and $\mathbf{q}$ as in Eq. (2.5) and Eq. (2.8), it can be concluded that the maximum values of each element in $\mathbf{R}$ and $\mathbf{q}$ should be 1 and -1. So the maximum values of attitudes can be used as a standard to evaluate the accuracy of each dynamical model with a special numerical algorithm.

7. Simulation analysis

In this Section, the eight dynamical models are solved and compared with the same numerical method. The length of simulation is 200 s, which reflects a long simulation time character. Supposing that the mass and inertia of the double pendulum is as follows: $m_1 = 0.5\,\mathrm{kg}$, $m_2 = 0.8\,\mathrm{kg}$, $J_1 = 1\,\mathrm{kg{\cdot}m^2}$,$J_2 = 2\,\mathrm{kg{\cdot}m^2}$, the length of the first pendulum is $l_1 = 1\,\mathrm{m}$. The position of mass center is $\boldsymbol{\rho}_1 = [0.05, 0.3]\,\mathrm{m}$, $\boldsymbol{\rho}_2 = [0.05, 0.3]\,\mathrm{m}$. The initial rotation angles and angular velocities of two stage pendulums are $\theta_1 = -\pi/3\,\mathrm{rad}$, $\theta_2 = \pi/3\,\mathrm{rad}$ and $\omega_1 = \omega_2 = 0\,\mathrm{rad/s}$, respectively. According to the initial angles, the initial rotation matrix and attitude can be derived from Eq. (2.5) and Eq. (2.8), respectively. The dynamical models are solved by the commonly used ordinary differential equation solver ODE45 in Matlab. So the accuracy of the eight models can be compared with the same algorithm. The simulation is performed by a computer with the Intel(R)Xeon(R) E-2176M CPU@2.70GHz processor, and the internal storage is 32GB.

The results of relative and absolute Lagrange dynamical equation based on the Lie group are denoted in Fig. 3 with black and red lines, respectively. The angular velocities and attitudes coincide in macroscopic scale. The simulation results distinguish from 50 s to 100 s and become bigger and bigger with time. The phase difference is more obvious than the amplitude, which means that the error accumulation have bigger influence on the phase than on the amplitude. A comparison of the Hamilton equations with the absolute and relative Lie group are shown in Fig. 4 with red and blue lines, respectively. The results are similar to Fig. 3, which indicates that the difference occurs at the time of 50 s.

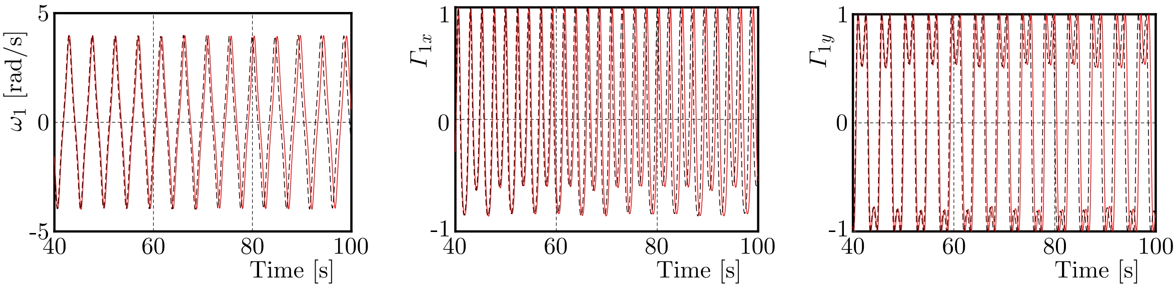

Fig. 3. A comparison of results for relative and absolute Lagrange equation based on the Lie group

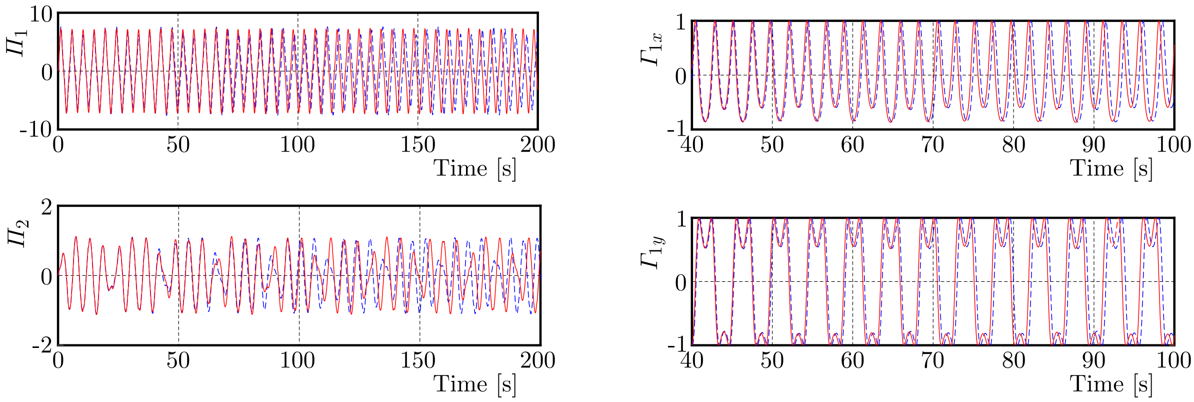

Fig. 4. A comparison of results for relative and absolute Hamilton equation based on the Lie group

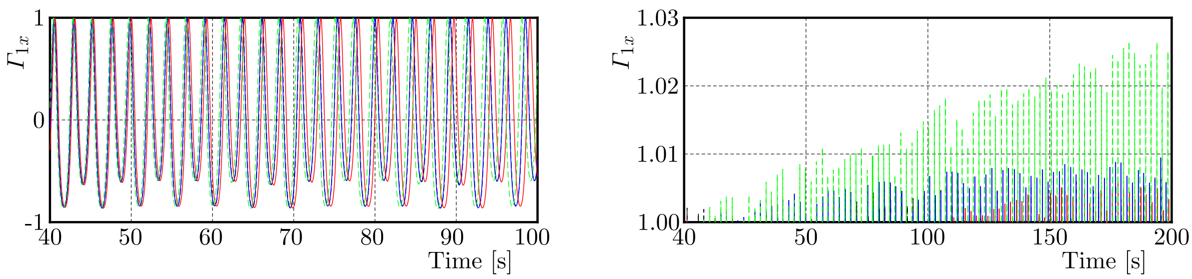

Fig. 5. A comparison of results of the relative/absolute Hamilton/Lagrange equation based on the Lie group

In Fig. 5, black and blue lines represent the results of relative motion, the red and green lines represent the absolute results. The results for relative motion coincide much more than the absolute ones. This means that the absolute modeling method will lead to fast error accumulation than the relative one. Cutting out the attitude values which are bigger than 1, the error accumulation of the relative group (black, blue) is smaller than 0.01 in 200 seconds. The results of the Lagrange equation for the absolute motion is also smaller than 0.01, and the beginning time of error accumulation is after 110 s. But the Hamilton equation for the absolute motion gives 0.03. So the computation result based on the Lagrange-Absolute-Lie group is advantageous obviously.

In Fig. 6, the green line represents the absolute modeling, and the red represents the relative one. The results start to distinguish at 50 s. In Fig. 7, the blue and black lines represent the absolute and relative results, respectively. They indicate that the momentum and attitudes are all out of sync for different models.

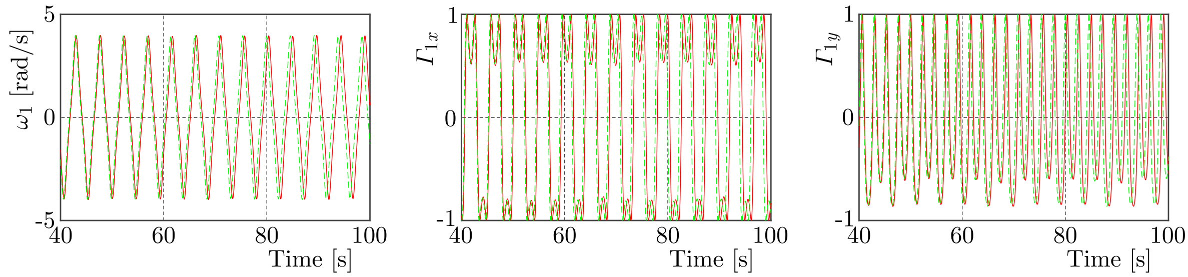

Fig. 6. Results of the Lagrange with the attitude vectors

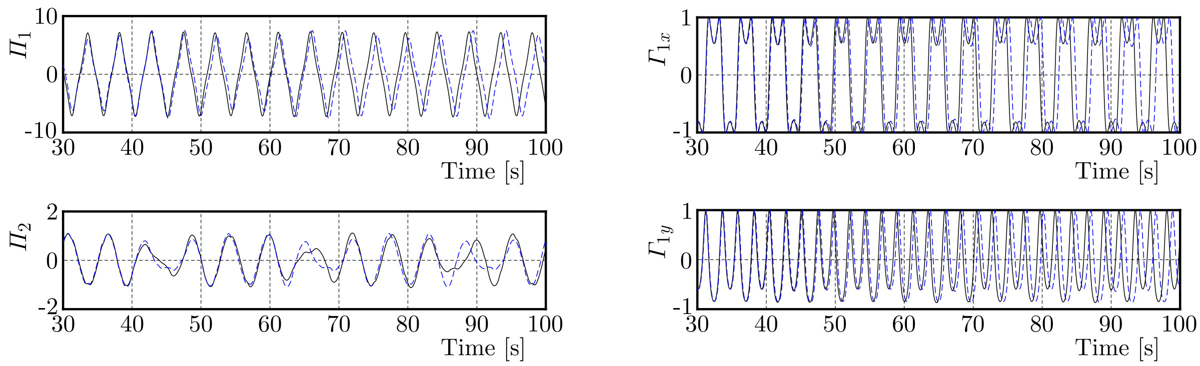

Fig. 7. Simulation results of the Hamilton equation with the attitude vector

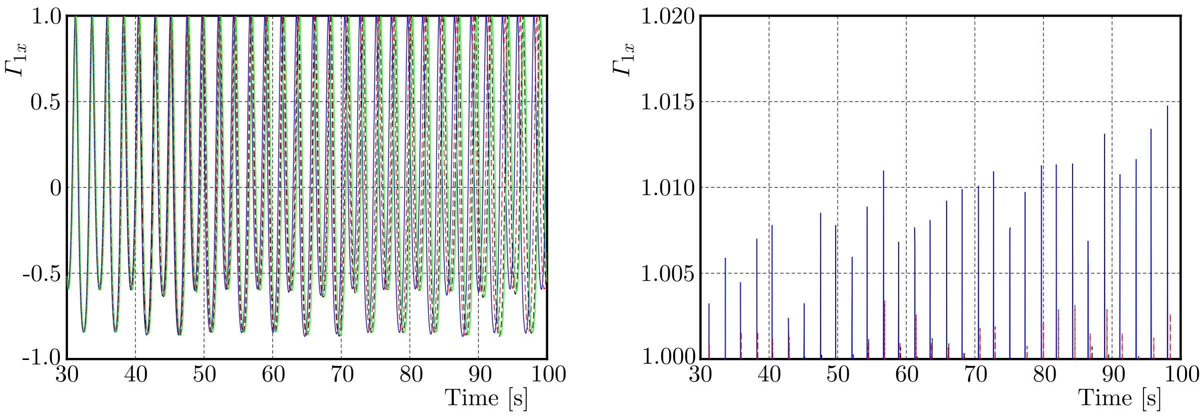

Fig. 8. A comparison of the first stage pendulum attitudes from four dynamics models based on the attitude vectors

According to Fig. 8, the four color lines exhibit a distinguishing tendency with a time increase. The black and green lines coincide much more than the other two from the results within 30 s to 100 s, which means that the results of the relative Hamilton equations coincide with the absolute Lagrange ones. The error represented by the blue line is much higher than others, which means that the error accumulation of the absolute Hamilton equation is much higher than the other three ones. The error accumulation of the absolute Lagrange equation begins at 110 s. The maximum value of the error is 0.015, which is lower than the Lie group result. In Fig. 9, three of these algorithms have the same long time simulation results. Only the Hamilton equation with the relative attitude vector has an obvious difference with respect to the other three algorithms. The error accumulation which is represented by the blue line is much higher than the other three ones, which means that the Hamilton equation with the absolute vector has faster error accumulation than the other three ones.

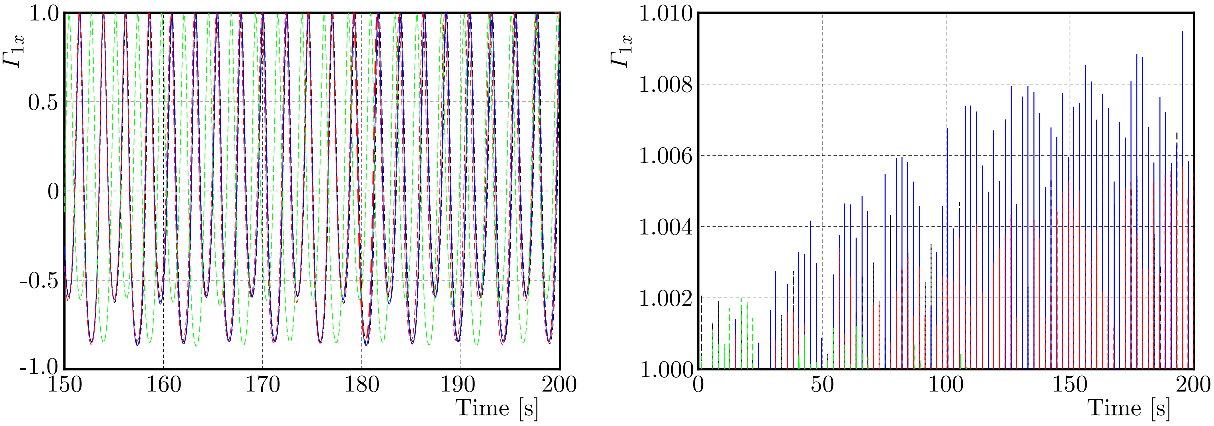

Fig. 9. A simulation results for the relative vector

In Fig. 10, the results coincide by pairs. The Lagrange and Hamilton functions coincide respectively. The error accumulation indicates that the Hamilton equation with the absolute attitude vector has the fastest error accumulation, and the maximum value is 0.02, which is higher than that with the relative vector (0.01).

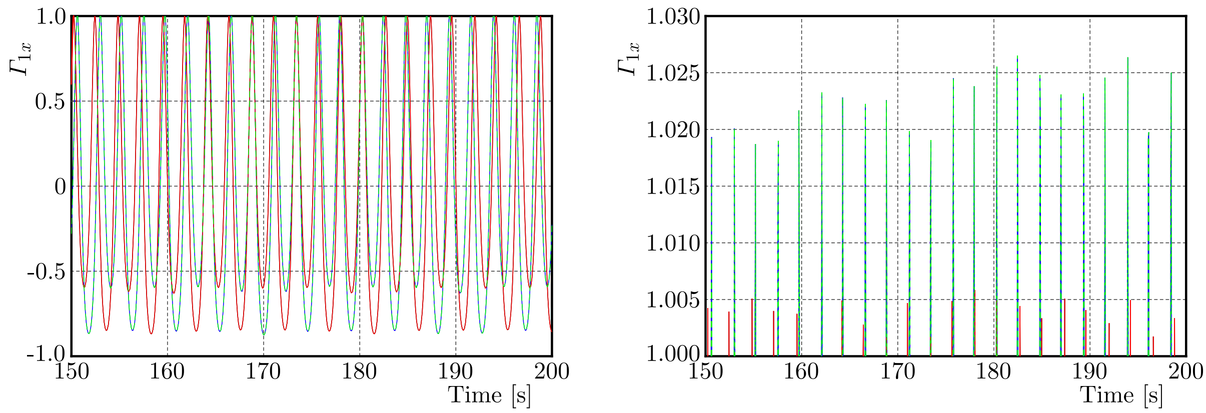

Fig. 10. Simulation results of the first stage pendulum with the under absolute vector

In Fig. 11, the green line differs more than the other three ones, which means that the Lagrange-Absolute-Attitude vector modeling method more easily leads to erratic results. The maximum attitude error is 0.006, which is the lowest one. In Fig. 12, there are three results from the four Hamilton methods. The green and red lines coincide together, which means that the results of Hamilton-Relative-Lie group and Hamilton-Relative-Attitude vector methods coincide together, which is better than for the two other methods. According to error accumulation, the Hamilton equation with the absolute vector has the biggest error. At 200 s of simulation, the error valued from 0.02 to 0.03.

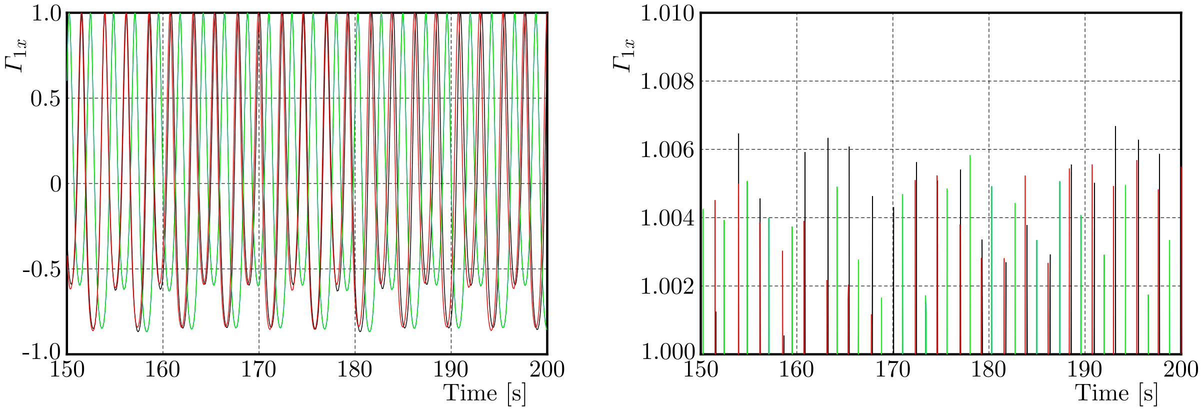

Fig. 11. Simulation results of the first stage pendulum from the Lagrange equation

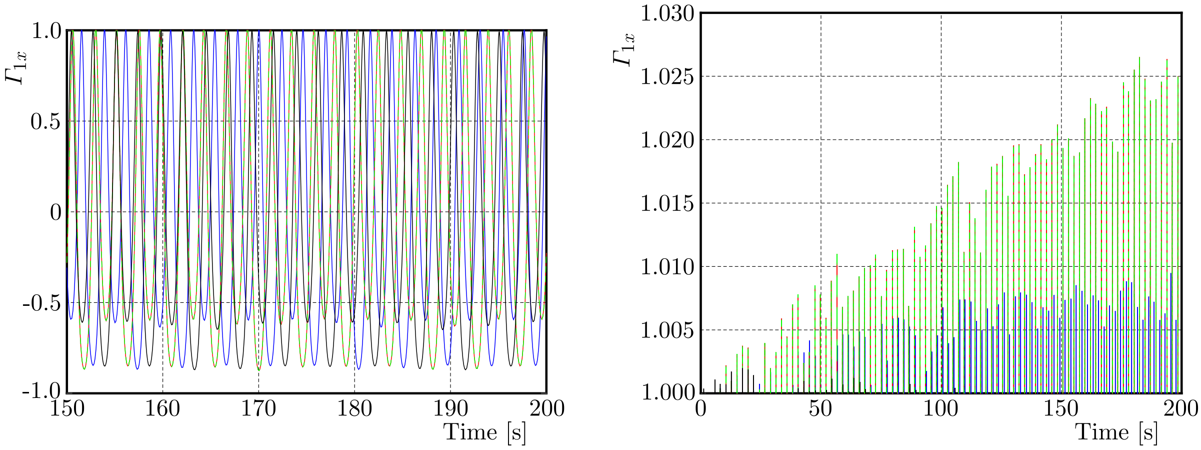

Fig. 12. Simulation results from the Hamilton equation

In Fig. 13, the red line (Lagrange equation for the absolute motion) represents the error occurring after 100 s, which has the lowest value. The green line (Hamilton equation for the absolute motion) represents the biggest error accumulation. The blue and cyan lines (relative motion) have approximate values. In the second one in Fig. 13, the green line (Absolute-Hamilton) coincides more with the red line (Absolute-Lagrange), whereas the blue and cyan lines (relative motion) obviously distinguish. This means that the absolute modeling method is more advantageous in conserving the character of long time simulation.

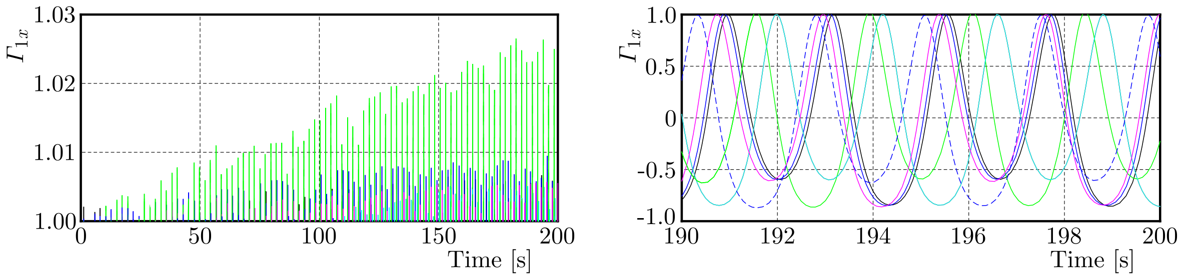

Fig. 13. Comparison of the eight methods

8. Conclusion

In this paper, geometry mathematical modeling (Lie group, attitude vector), dynamical modeling method (Lagrange and Hamilton) and motion modeling (absolute and relative motion) are all combined together, and eight dynamical equations of a double pendulum are built. The equations are solved by the same numerical algorithms. According to comparison of the resuts, the Lagrange equation with the absolute Lie group has the best conservation character in a long time simulation. From complexity of the model, the Hamilton equation for the absolute motion has the simplest structure. So the geometry dynamical models of the Hamilton type and with the absolute motion are advantageous in both the modeling as well as computations.

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References

1. Arnold M., Brüls O., Cardona A., 2015, Error analysis of generalized-α Lie group time integration methods for constrained mechanical systems, *Numerische Mathematik*, **129**, 149-179
2. Bjorkenstam S., Leyendecker S., Linn J., Carlson J.S., Lennartson B., 2018, Inverse dynamics for discrete geometric mechanics of multibody systems with application to direct optimal control, *Journal of Computational and Nonlinear Dynamics*, **13**, 101001-1
3. Celledoni E., Çokaj E., Leone A., Murari D., Owren B., 2021, Lie group integrators for mechanical systems, *International Journal of Computer Mathematics*, **99**, 1, 58-88
4. Ding J., Pan Z., 2014, Higher order variational integrators for multibody system dynamics with constraints, *Advances in Mechanical Engineering*, **2014**, ID 383680
5. Ding J.Y., Pan Z.K., Zhang, 2019, The constraint- stabilized implicit methods on Lie group for differential-algebraic equations of multibody system dynamics, *Advances in Mechanical Engineering*, **11**, 4
6. Holzinger S., Gerstmayr J., 2021, Time integration of rigid bodies modelled with three rotation parameters, *Multibody System Dynamics*, **53**, 345-378
7. Lee T., Leok M., McClamroch N.H., 2009, Lagrangian mechanics and variational integrator on two-spheres, *International Journal for Numerical Methods in Engineering*, **79**, 9, 1147-1174
8. Müller A., 2014, Higher derivatives of the kinematic mapping and some applications, *Mechanism and Machine Theory*, **76**, 70-85
9. Müller A., 2021, Review of the exponential and Cayley map on SE(3) as relevant for Lie group integration of the generalized Poisson equation and flexible multibody systems, *Proceedings of the Royal Society of London. Series A*, **477**, 20210303
10. Müller A., 2022, Dynamics of parallel manipulators with hybrid complex limbs-Modular modeling and parallel computing, *Mechanism and Machine Theory*, **167**, 104549
11. Müller A., Terze Z., 2014, The significance of the configuration space Lie group for the constraint satisfaction in numerical time integration of multibody systems, *Mechanism and Machine Theory*, **82**, 173-202
12. Rong J., Wu Z., Liua C., Brüls O., 2020, Geometrically exact thin-walled beam including warping formulated on the special Euclidean group SE(3), *Computer Methods in Applied Mechanics and Engineering*, **369**, 113062

13. Sonneville V., Bruls O., 2014, Sensitivity analysis for multibody systems formulated on a Lie group, *Multibody System Dynamcis*, **31**, 1, 47-67

14. Terze Z., Müller A., Zlatar D., 2015, Lie-group integration method for constrained multibody systems in state space, *Multibody System Dynamics*, **34**, 275-305

15. Urkullu G., de Bustos I.F., Garca-Marina V., Uriarte H., 2019, Direct integration of the equations of multibody dynamics using central differences and linearization, *Mechanism and Machine Theory*, **133**, 432-458

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