JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
60, 3, pp. 509-520, Warsaw 2022
https://doi.org/10.15632/jtam-pl/151947

LAMINAR FLOW PAST THE BOTTOM WITH OBSTACLES – A POROUS MEDIUM APPROXIMATION

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We provide a model of a stationary laminar flow in a channel at the bottom of which plants grow in a dense layer. Since this layer of plants is dense, we treat it as a porous medium and we propose to describe the flow in such a medium by Brinkman's equation. The flow in the fluid layer located above (infiltrated by water) the layers of plants is described by the Stokes equation. We show that such a model gives results consistent with experimental observations. We indicate also that this new model complements the previously given model in which the benthonic plants were considered as a suspension, so the previous model referred to the channel at the bottom of which the plants grew rare. For high permeability of the porous medium, we arrive at the results obtained for the medium filled with a liquid with a suspension.

Keywords: plants, permeability, gravity driven flow, Darcy-Brinkman's law, Stokes equation

1. Introduction

The aim of the study is to provide a new description of the steady flow of a liquid in the channel at the bottom with obstructions, in particular bottom plants. The novelty of our description consists in replacing the bottom obstacles with a porous medium in which the liquid flow is described by the Brinkman equation.

Even before the last two decades, the problem of flow in a channel with a bottom vegetation was regarded as an open task. First, it is worth to mention a book on asymptotic behaviour of solutions to boundary value problems at rough edges (Marchenko and Khruslov, 1974). Applying a certain averaging method (type of homogenization), these authors essentially derive the stationary Brinkman equation. One of the proposed methods was the double averaging method (after space and time) introduced as natural extensions of single averaged Reynolds equations (Nikora *et al.*, 2001). The averaging method replaces the turbulence field with an additional member – the turbulence tensor.

The problem of flow through an overgrown channel is related to the problem of flow over an uneven bottom, which has recently been discussed by many authors (Malevich *et al.*, 2006).

The flow in the channel with a bottom bed vegetation was considered as a turbulent flow, and it was pointed out that the flow velocity distribution with the height of the liquid column has a shape described by tanh function (Nepf, 2012).

The mutual influence of the bottom plant deformation and the water current was accounted for by Kubrak *et al.* (2015), Tang (2019) and D'Ippolito *et al.* (2019, 2021).

The problem of flow in a channel with plants on the bottom was solved as a task of two--component flow. It was pointed out that there exists a possibility of appearing vortices both in the bottom area occupied by plants, and at the interface of the plant canopy and free flow (Yang and Choi, 2009, 2010; Huai *et al.*, 2013). This approach boils down to the acceptance of viscosity increased by the effect of turbulence in the canopy region.

Experimental studies of flows in a channel with a porous bottom in the aspect of the Brinkman equation were carried out by Morad and Khalili (2009).

In our work, we also treat the flow in two layers, denoted by A and B. We propose a model of a stationary laminar flow in the channel at the bottom of which plants grow in a dense layer. This bottom area is denoted by the letter B, and it is treated as a porous medium. We admit that the flow in such a medium is described by Brinkman's equation. The flow in the fluid layer located above (layer A) is described by Stokes' equation (Wojnar and Bielski, 2015).

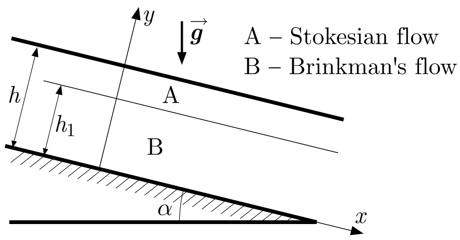

Fig. 1. Flow of the stream is developing in two manners depending on which regions of A or B, is occupied. In region A the flow is laminar, while in region B we deal with a seepage through the porous medium. The gravity acceleration vector **g** has two components $[g_x, g_y] = [g\sin\alpha, -g\cos\alpha]$

At the interface of A and B layers, we assume continuity of velocity and continuity of tangential stresses. Therefore, we omit possibilities of turbulence at the interface. The vertical distribution of velocity in our B layer region is described by sinh function, and by a parabola in A layer region.

If the asperities, stones, plants or other obstacles are small, in the limiting case we arrive at the concept of the rough bottom. In particular, it is a case in lakes, streams, rivers and open channels, in which different types of plants may grow (Yang and Choi, 2009, 2010; Huai *et al.*, 2013; Evangelos, 2012).

The asymptotic analysis was applied to the flow over the rough bottom in the paper by Bielski and Wojnar (2021).

Firstly, in our study we present Brinkman's equation (Section 2) and Stokes' equation (Sections 3 and 4), which will be applied in the paper. The solutions to Brinkman's equation for special problems are given in Sections 5 and 6, where numerical examples are presented.

Section 7 compares our results with the experimental evidence obtained in (Kubrak *et al.*, 2015; Caroppi *et al.*, 2019; Pu *et al.*, 2019). For illustration, a two-dimensional but one-directional flow past the plane covered with obstacles is considered, and the results of analytical calculations are presented for both ways of approximation. On the interface, continuity of velocity and shear stress is postulated.

It will be shown that it is consistent with the experiment to assume continuity of velocity and shear stresses at the boundary of both media (porous, benthic plants with the fluid, medium B) and the free fluid (medium A).

In Section 8 a proof is given that solution (7.6) obtained for the porous medium in Section 5 passes gently in the limit of high permeability into a solution for suspension, given by Eq. (21) in (Wojnar and Bielski, 2015). In such a manner, this new model complements the previously proposed one, in which bottom plants were considered as a suspension.

2. Darcy's and Brinkman's laws

Darcy's law was formulated on the basis of experimental studies of water seepage in sand columns. In the contemporary formulation, it states a linear relation between the velocity $\mathbf{v}$ and the sum of the gradient of pressure ∇p and the body force $\boldsymbol{\gamma}$. It summarizes properties exhibited by the ground water flowing in the bulk of aquifers. However, it cannot be used to account for transitional flow between boundaries of a porous medium and the Stokes flow beyond this medium. Experimental tests have shown that flow regimes with Reynolds numbers up to 10 may still be Darcian, as in the case of the ground water flow. Darcy's law can be derived from Stokes' equation of a viscous fluid flow *via* homogenisation (Auriault, 2009; Bielski and Wojnar, 2008).

Henri Coenraad Brinkman proposed an equation describing seepage of the fluid through a porous medium in a more adequate way than Darcy's law does, cf. Brinkman (1949), Auriault (2009), Joodi *et al.* (2010). The equation gives the following dependence between the fluid velocity $\mathbf{v}$ and the sum of the pressure gradient ∇p and the body force vector $\boldsymbol{\gamma} = \rho\mathbf{g}$

$$\mathbf{v} = \frac{K}{\eta}(-\nabla p + \boldsymbol{\gamma}) + K\frac{\eta'}{\eta}\Delta\mathbf{v} \tag{2.1}$$

Here p denotes the pressure and K is the permeability of the porous medium. The coefficient η is the fluid viscosity and the coefficient η' (known also as the effective viscosity) is a modified fluid viscosity which may be different from η. Both coefficients, η and η', are assumed to be constant. The coefficient η', Eq. (2.1), is a measure of significance of the Stokesian part in Brinkman's equation. The η' coefficient is estimated experimentally. Some computational evaluation of η' was given by Valdes-Parada *et al.* (2007).

Far from the boundaries of the porous medium, in the bulk, where the values of velocity gradients are low, the Laplacian $\Delta\mathbf{v}$ can be neglected, and Eq. (2.1) is approximated by Darcy's equation

$$\mathbf{v} = \frac{K}{\eta}(-\nabla p + \boldsymbol{\gamma}) \tag{2.2}$$

For high values of K, Stokes' equation (it is Navier-Stokes' equation with the inertial terms neglected) is obtained

$$-\nabla p + \boldsymbol{\gamma} = \eta\Delta\mathbf{v} \tag{2.3}$$

If we apply the operator ∇ to Eq. (2.1), and if the fluid is incompressible

$$\nabla \cdot \mathbf{v} = 0 \tag{2.4}$$

then we receive $\nabla \cdot (-\nabla p + \boldsymbol{\gamma}) = 0$.

3. Stokes' equation

We consider the flow of an incompressible viscous fluid of density ρ and viscosity η, and examine a gravity driven steady flow of this fluid. The velocity field ruling in the fluid is given by the vector $\mathbf{v} = \mathbf{v}(\mathbf{x})$, with $\mathbf{x} = [x_1, x_2, x_3]$.

Navier-Stokes' equation of steady motion of an incompressible fluid with constant viscosity η under the pressure gradient ∇p and the gravitation force $\boldsymbol{\gamma} = (\gamma_i)$, $i = 1, 2, 3$, reads

$$\rho v_k \frac{\partial v_i}{\partial x_k} = \gamma_i - \frac{\partial p}{\partial x_i} + \eta\frac{\partial^2 v_i}{\partial x_j \partial x_j} \tag{3.1}$$

We admit that the body force $\boldsymbol{\gamma}$ does not depend on the position $\mathbf{x}$. If the velocity has only one component, say v_1, it is when

$$\mathbf{v} = [v_1, 0, 0] \tag{3.2}$$

and this component does not depend on x_1 and x_3, it is $v_1 = v_1(x_2)$, then the left hand side of Eq.(3.1) vanishes and the equation becomes, cf. Eq. (2.3)

$$\gamma_i - \frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j} = 0 \tag{3.3}$$

This is known as *Stokes' equation.* In an explicit form

$$\gamma_1 - \frac{\partial p}{\partial x_1} + \eta\Big(\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_3^2}\Big) = 0 \qquad\qquad \gamma_2 - \frac{\partial p}{\partial x_2} = 0 \tag{3.4}$$

For brevity, we are omitting subscript 1 and put simply $v_1 \equiv v$.

In Eq. (3.4), the pressure p does not depend on x_3 and the component γ_3 vanishes.

4. Flow of the fluid with the free upper surface

We consider the flow of the fluid on an inclined plane under the influence of gravity, see Fig. 1. Thus, we discuss only a central part of the fluid stream and neglect the influence of the stream banks, so the problem is two-dimensional.

We change the notation, and substitute

$$x_1 = x \qquad x_2 = y \qquad x_3 = z \qquad \text{and} \qquad \boldsymbol{\gamma} = [\gamma_x, -\gamma_y, 0]$$

Let a layer of an incompressible viscous fluid of total thickness h be bounded above by a free surface and below by a fixed plane inclined at an angle α to the horizontal, see Fig. 1. Let us determine the steady flow due to gravity when the flow in the upper part (region A) is free and in the lower part (region B) it passes through a porous medium.

We take the fixed plane as the xz-plane, with the x-axis in the direction of flow and y pointed upward in the direction perpendicular to the bottom. Let the viscosity be a function of y only and we seek for a solution depending only on y. Stokes' equations (3.4) with $v_1 = v(y)$ in a gravitational field reduce to two equations

$$\eta \frac{d^2 v}{dy^2} + \gamma_x = 0 \qquad\qquad \frac{dp}{dy} + \gamma_y = 0 \tag{4.1}$$

Here (see Fig. 1)

$$\gamma_x = \rho g \sin\alpha \qquad\qquad \gamma_y = -\rho g \cos\alpha \tag{4.2}$$

and g denotes the value of gravitational acceleration.

5. Flow past porous medium

We refer again to Fig. 1. The space accessible to the fluid is divided in two regions, A and B. In each region the flow of the same fluid (with viscosity η) is developing in two different manners: in upper region A (i.e., for $h_1 < y < h$), the flow is Stokesian, and in lower region B (for $0 \leqslant y < h_1$) we are dealing with the seepage through a porous medium.

5.1. Flow in the region A

We retain notation of components introduced in the previous Section. From the first equation of the system (4.1), we get after the second integration

$$v(y) - v_B = \frac{1}{\eta}\Big(-\frac{1}{2}\gamma_x y^2 + Cy\Big) \tag{5.1}$$

The constants v_B and C are to be found from the boundary conditions.

Integration of the second equation of the system (4.1) gives

$$p(y) = -\gamma_y y + p_0 \tag{5.2}$$

where p_0 is a constant.

At the free surface ($y = h$), where the atmospheric pressure p_A prevails, we have $p = p_A$. Hence $p_0 = p_A + \gamma_y h$, and for $h > y > h_1$, one obtains

$$p = (h - y)\gamma_y + p_A \tag{5.3}$$

At the free surface also $\sigma_{xy} = 0$

$$\sigma_{xy}(h) = \Big\{\eta\frac{dv}{dy}\Big\}_{y=h} = 0$$

and after Eq. (5.1)

$$C = \gamma_x h \tag{5.4}$$

At the bottom of region A, it is for $y = h_1$, in other words at *the interface* of regions A and B, the fluid velocity $v(h_1) \equiv v_{int}$ is still unknown. The integration constant v_{int} will be found from comparison with the solution in lower porous region B. Now, for the region $h > y > h_1$, we write only

$$v(y) = \frac{\gamma_x}{\eta}\Big(h - \frac{1}{2}y\Big)y + v_{int} \tag{5.5}$$

This is a formal solution for region A ($h_1 < y < h$) in which the constant v_{int} is unknown.

The shear component in the flow in region A is a linear function of y and reads

$$\sigma_{xy} = \gamma_x(h - y) \tag{5.6}$$

This is the shear distribution in region A.

5.2. Flow in lower region B

We assume, as in the previous Section that $\partial p/\partial x_1 = 0$, cf. Eq. (5.3), and apply Brinkman's equation (3.1) for the study of flow in the gravitational field. As in the previous Section, we assume also that just the component $v_x \equiv v$ does not vanish, and it depends on the y variable only. In view of these assumptions, Brinkman's equation transforms into

$$\frac{d^2v}{dy^2} - \frac{\eta}{\eta' K}v + \frac{\gamma_x}{\eta'} = 0 \tag{5.7}$$

Its general solution is

$$v = b_1 e^{ay} + b_2 e^{-ay} + \frac{K}{\eta}\gamma_x \tag{5.8}$$

where

$$a = \sqrt{\frac{\eta}{\eta' K}} \tag{5.9}$$

while b_1 and b_2 are constants to be found.

Since the velocity v at the bottom ($y = 0$) should vanish, we have from Eq. (5.8)

$$b_1 + b_2 = -\frac{K}{\eta}\gamma_x \tag{5.10}$$

Now, three constants, b_1, b_2 and v_{int} are to be found, cf. Eqs. (5.9) and (5.8).

5.3. Interface between the free flow (A) and seepage (B) regions

At the interface between regions A and B, the continuity of velocities and shear stresses should be assured. The continuity of velocities at $y = h_1$ requests that

$$v_{int} = b_1 e^{ah_1} + b_2 e^{-ah_1} + \frac{K}{\eta}\gamma_x - \frac{\gamma_x}{\eta}\Big(h - \frac{1}{2}h_1\Big)h_1 \tag{5.11}$$

The continuity of shear stresses means that

$$g_x(h - h_1) = a\eta' b_1\Big(e^{ah_1} + e^{-ah_1}\Big) + a\eta'\frac{K}{\eta}\gamma_x e^{-ah_1} \tag{5.12}$$

since by Eq. (5.10) we have $b_2 = -(K/\eta)\gamma_x - b_1$. Finally, by Eq. (5.12)

$$b_1 = \frac{h - h_1 - a\eta'\frac{K}{\eta}e^{-ah_1}}{a\eta'(e^{ah_1} + e^{-ah_1})}\gamma_x \tag{5.13}$$

In this manner, the explicit form of our solution reads

$$v = \begin{cases} \dfrac{\gamma_x}{\eta}\Big(h - \dfrac{1}{2}y\Big)y + v_{int} & \text{in A} \\ \Big(\dfrac{h - h_1}{a\eta'} - \dfrac{K}{\eta}e^{-ah_1}\Big)\gamma_x\dfrac{e^{ay} - e^{-ay}}{e^{ah_1} + e^{-ah_1}} + \dfrac{K}{\eta}\gamma_x(1 - e^{-ay}) & \text{in B} \end{cases} \tag{5.14}$$

where

$$v_{int} = \Big(\frac{h - h_1}{a\eta'} - \frac{K}{\eta}e^{-ah_1}\Big)\gamma_x\frac{e^{ah_1} - e^{-ah_1}}{e^{ah_1} + e^{-ah_1}} + \frac{K}{\eta}\gamma_x(1 - e^{-ah_1}) - \frac{\gamma_x}{\eta}\Big(h - \frac{1}{2}h_1\Big)h_1$$

and a is given by formula (5.9).

6. Calculatory examples

Relation (5.14) was applied in the following examples.

The flow in the unit depth, $0 < y \leqslant h = 1$, partially through the porous medium at the bottom, $0 < y \leqslant h/2 = 1/2$, is considered. The calculations are performed for the parameters given in the natural units, in which the value of volume force $\gamma_x = 1$, see Eq. $(4.2)_1$, and the value of viscosity $\eta = 1$, see below the relations (7.2) and (7.3). The results for different permeabilities κ are shown in Figs. 2-4. The permeability κ given in the natural units is related to the permeability K given in the metrical system by Eq. (7.1).

In Fig. 2, we present the velocity v as a function of the space variable y for different permeabilities, beginning from the lowest $\kappa = 0.001$ to the highest $\kappa = 0.015$. The effective viscosity κ'

is equal to the viscosity κ in this case. The natural system of units defined by (7.2) and (7.3) are used. For values $\kappa = 0.001$ and $\kappa = 0.002$, the flow in the bulk of the porous medium is constant and is near to Darcy's law. We observe that the velocity v in the bulk porous medium is almost constant except for the regions nearest to the boundaries, $y = 0$ and $y = 1/2$. At these boundaries the velocity changes abruptly but continuously. In this case, the velocity in the bulk of the porous medium does not depend on the height y, as is provided by Darcy's law (3.2). Namely, according to this law for $\nabla p = \mathbf{0}$, as is in our case, the velocity v is proportional to the given volume force γ_x.

The region of the abrupt change is determined by the exponent coefficient a, Eq. (5.9). With the increasing permeability κ, this coefficient decreases and the region of the velocity change becomes larger, as observed in Fig. 2 for $\kappa = 0.010$ and $\kappa = 0.015$.

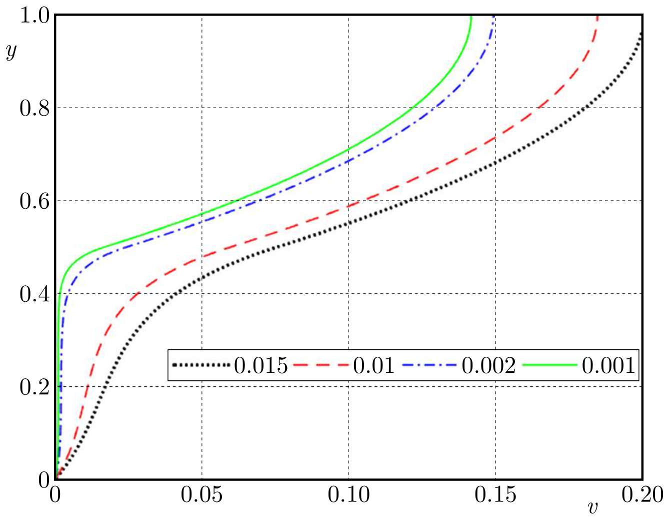

Fig. 2. Dependence of the fluid velocity v on the height y for growing permeabilities κ equal respectively to 0.001, 0.002, 0.010, and 0.015 (in natural units). In this case, we accept $\kappa = \kappa'$

7. Relation to the experiment

We verify our proposal with an experiment performed by Kubrak *et al.* (2013). To this end, we solve the problem formulated in Table 1, where

Table 1. Darcy's description of the flow past bottom plants

Range	Equation	Solution
$0 < y < 0.1$	$d^2v/dy^2 = -1$	$v = -0.5y^2 + Cy$
$0.1 < y < 0.35$	$v = \kappa$	$v = \kappa$
$0.35 < y < 1$	$d^2v/dy^2 = -1$	$v = (1 - 0.5y)y + A$

$$C = \frac{\kappa}{h_1} + \frac{1}{2}h_1 \qquad A = \kappa - \Big(1 - \frac{1}{2}h_2\Big)h_2$$

with $h_1 = 0.1$ and $h_2 = 0.35$.

We observe in Table 1 that in regions $0 < y < 0.1$ and $0.35 < y < 1$, the free flow is admitted, and in $0.1 < y < 0.35$ the Darcy flow occurs.

In Fig. 3, we compare our solution of Darcy's problem with the velocity distribution calculated in Kubrak *et al.* (2012). The experimental points were measured at different sections of the flowing stream and were significantly spread. Only one section of experimental data is in close agreement with the theory of Kubrak *et al.* (2013) basing on the analysis of deflection of individual stems. We observe that this theoretical line is in accord with the velocity distribution given by Darcy's equation. In general fitting of experimental data, Brinkman's distribution of velocity is more consistent than Darcy's description, and fits better with the experimental results (Kubrak *et al.*, 2013).

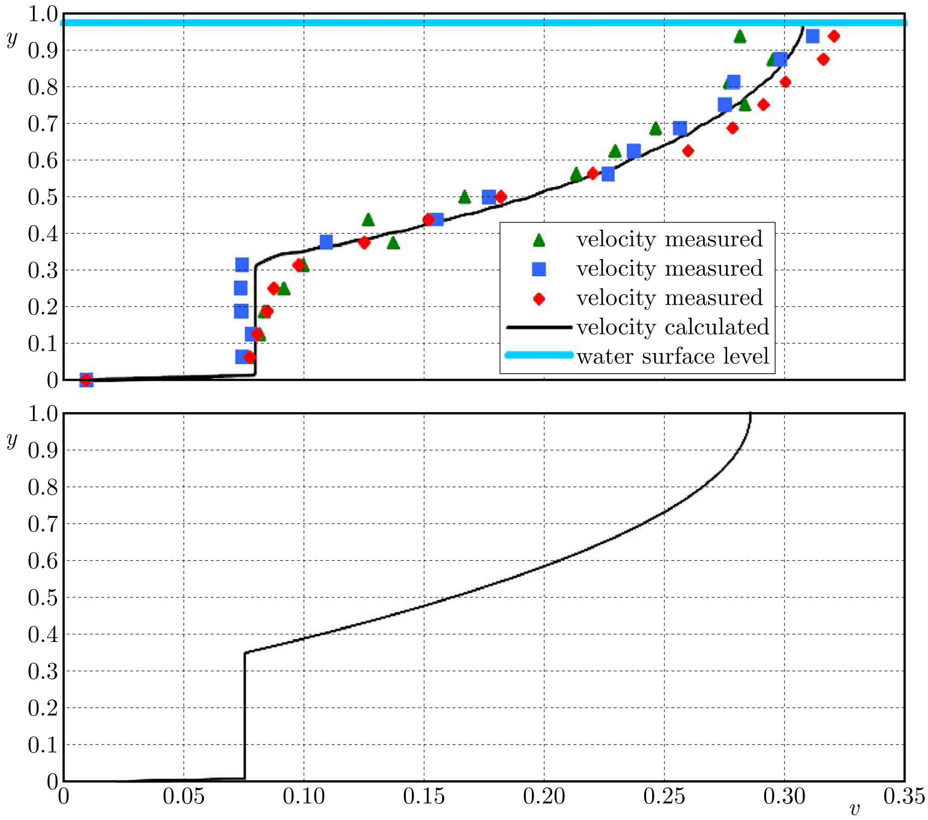

Fig. 3. Comparison of experimental results and the solution given by Kubrak *et al.* (2013) (upper) with description obtained by Darcy's law (lower). The velocity scale on horizontal axes in both figures are the same

In Fig. 4, a family of velocity distributions obtained from Brinkman's equation is revealed. In this figure, the permeability κ is the same for all curves but the permeability

$$\kappa' = \kappa\frac{\eta'}{\eta} \tag{7.1}$$

is varied.

From two Brinkman's viscosity coefficients, the first η is the same for all the curves but the second viscosity coefficient η' is varied. By a suitable choice of η' (fitted to the experimental conditions) we can describe behaviour of velocity distribution in the given section of the fluid flow. Darcy's flow given in Table 1 is a limiting process for considered Brinkman's flows (it gives the upper bound of velocities for small velocities, $v < \kappa$, and the lower bound for velocities $v > \kappa$). The natural unit of pressure $\breve{p}$ is the x component of the hydrostatic pressure at the bottom, which assures the unit value of the x component of the body force. Thus

$$\breve{x} = h \qquad \breve{v} = 3v^{mean} = \frac{\rho}{\eta}h^2 g\sin\alpha \qquad \breve{p} = \frac{\eta}{h}3v^{mean} = \rho hg\sin\alpha \tag{7.2}$$

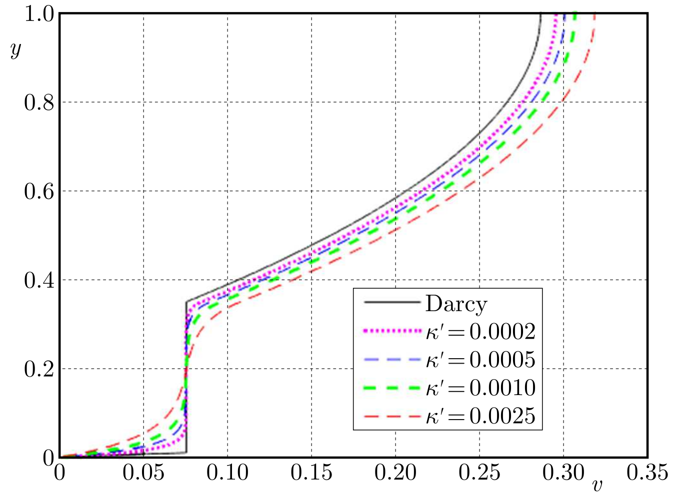

Fig. 4. Dependence of the fluid velocity v on the height y for a given permeability $\kappa = 0.075$ and for different effective permeabilities κ' equal to 0.0002, 0.0005, 0.001 and 0.0025 (in natural units). The behaviour of velocity given by Darcy's equation, cf. Table 1, is given by the continuous line

Consequently, the unit of time is

$$\breve{t} = \frac{\breve{x}}{\breve{v}} = \frac{h}{3v^{mean}} = \frac{\eta}{\rho h g \sin\alpha} \tag{7.3}$$

Then, new variables $\widetilde{x}$, $\widetilde{v}$ and $\widetilde{p}$ are introduced: $\widetilde{x} \equiv x/\breve{x}$, $\widetilde{v} \equiv v/\breve{v}$ and $\widetilde{p} \equiv p/\breve{p}$.

In the new variables, Stokes' and Brinkman's equations have the form

$$\frac{d^2\widetilde{v}}{d\widetilde{y}^2} + 1 = 0 \qquad\qquad \frac{d\widetilde{p}}{d\widetilde{y}} + \cot\alpha = 0 \tag{7.4}$$

and

$$\widetilde{v} = \kappa\Big(-\frac{\partial\widetilde{p}}{\partial\widetilde{x}} + 1\Big) + \kappa'\Big(\frac{\partial^2\widetilde{v}}{\partial\widetilde{x}^2} + \frac{\partial^2\widetilde{v}}{\partial\widetilde{y}^2}\Big) \tag{7.5}$$

where

$$\kappa \equiv \frac{K}{h^2} \qquad\qquad \kappa' \equiv \frac{K}{h^2}\frac{\eta'}{\eta} \tag{7.6}$$

8. Limit passage to high permeabilities

It would be of interest to study our solution for high values of the permeability K.

For small values of the coefficient a, Eq. (5.9), that is for high values of the permeability K, Brinkman's equation (3.1) transforms into Stokes' equation (3.3).

We ask whether for small values of a it is possible to regain relation (5.11) of the previous paper (Wojnar and Bielski, 2015) from solution (5.14) in the present paper.

We recall, for compactness of the derivation, that Eq. (21) of the paper (Wojnar and Bielski, 2015) in the notation of the present paper reads

$$v(y) = \frac{1}{\eta}\gamma_x\Big(h - \frac{1}{2}y\Big)y$$

In the present problem solution (5.14), for small a we develop exponentials in Taylor's series and receive

$$v = b_1\Big(1 + ay + \frac{1}{2}a^2y^2 + \cdots\Big) + b_2\Big(1 - ay + \frac{1}{2}a^2y^2 + \cdots\Big) + \frac{K}{\eta}\gamma_x$$

or

$$v = b_1 + b_2 + (b_1 - b_2)ay + (b_1 + b_2)\frac{1}{2}a^2y^2 + \frac{K}{\eta}\gamma_x \tag{8.1}$$

if powers of a higher than 2 are rejected. Firstly, we observe that in virtue of Eq. (5.10) the sum $b_1 + b_2 + (K/\eta)\gamma_x$ vanishes and the constant term in Eq. (5.11) is equal to zero, similarly as in Eq. (21) of our previous paper (Wojnar and Bielski, 2015).

Next, we find the coefficient at y^2. We use Eq. (5.10) and get

$$(b_1 + b_2)\frac{1}{2}a^2 = -\frac{K}{\eta}g_x\frac{1}{2}a^2 = -\frac{1}{2}\frac{\gamma_x}{\eta'}$$

Here, also relation (5.9) was used. This result is identical with the coefficient at y^2 in Eq. (21) of the paper (Wojnar and Bielski, 2015), if only η' is substituted by η.

The coefficient at the linear term in Eq. (8.1) reads $(b_1 - b_2)a$, or accounting for Eq. (5.10)

$$\Big(2b_1 + \frac{K}{\eta}\gamma_x\Big)a$$

For small a, if exponentials in b_1 are to be linearized

$$\Big(2\frac{h - h_1 - a\eta'\frac{K}{\eta}(1 - ah_1)}{a\eta'(1 + ah_1 + 1 - ah_1)}\gamma_x + \frac{K}{\eta}\gamma_x\Big)a$$

we get

$$\frac{h - h_1}{\eta'}\gamma_x - a\frac{K}{\eta}(1 - ah_1)\gamma_x + \frac{K}{\eta}\gamma_x a = \frac{h}{\eta'}\gamma_x$$

If only η' and η are identified, one obtains the linear coefficient of flow described by Eq. (21) of the paper (Wojnar and Bielski, 2015). Therefore, the asymptotic behaviour (as $K \to \infty$) of our solution for the porous medium is that of the suspension flow.

9. Conclusions

The aim of our work was to provide a simple way of describing flows in channels with an overgrown bottom. We chose an analytical method based on Brinkman's law, wanting to provide a specific simple way of describing such a flow. Bottom flow problems can be solved by Computational Fluid Dynamics methods, in which different porosities of the bottom can be taken into account. However, in this way we get numerical results without being able to get a general view of the problem. It is fortunate that our simple analytical description captures the main features of experimental measurements. Our results are consistent with the experimental data reported in the work of Kubrak *et al.* ((2012, 2013) and Yang and Choi (2009, 2010).

We regard that such a system permits one to study characteristic traits of the flows in beds of rivers (canals, pipes, lakes) with obstacles at the bottom (such as stones, plants or other structures), which are not susceptible to outer influences.

This issue has been studied for over two decades and our contribution is to introduce Brinkman's flow to the description of flows through obstacles.

Above, we have introduced the porous medium to simulate the obstacles met by the flowing water in river beds, and applied Brinkman's equation to that.

- We have applied Brinkman's equation to describe a flow through a porous medium at the canal bottom.
- We have demonstrated that for low permeabilities, the flow in the bulk of a porous medium is Darcy's flow.
- For high permeabilities, the influence of boundaries (described by the term of Brinkman) becomes visible and the flow resembles the flow of suspensions discussed in (Wojnar and Bielski, 2015). This also finds an expression in the limit passage to high permeabilities (cf. Section 8). However, quantitative differences are significant since in the present solution a sudden, albeit continuous change of the fluid velocity on the porous medium, a free fluid interface is observed.
- It is worth noting that from the qualitative point of view our solution is similar to the solution in (Nepf, 2012). We would like to draw the attention to an appropriate match of our results with experimental data in (Kubrak *et al.*, 2013).

Comparing quantitatively and qualitatively our calculations with the experimental results obtained by Kubrak *et al.* (2013), we have found that by a suitable choice of material parameters we can model the real flows past the bottom with obstacles within a wide range of concentrations. Significance of the parameter called by Brinkman as the *effective viscosity coefficient* η' has been demonstrated.

Acknowledgements
This work was partially supported within statutory activities No. 3841/E-41/S/2020.

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Manuscript received September 17, 2021; accepted for print June 20, 2022