JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
59, 3, pp. 443-454, Warsaw 2021
https://doi.org/10.15632/jtam-pl/138855

LONG GRAVITY WAVES IN A CANAL WITH A CORRUGATED BOTTOM IN THE ASYMPTOTIC DESCRIPTION

Włodzimierz Bielski
Institute of Geophysics, Polish Academy of Sciences, Warsaw, Poland
e-mail: wbielski@igf.edu.pl

Ryszard Wojnar
Institute of Fundamental Technological Research, PAS, Warsaw, Poland
e-mail: rwojnar@ippt.pan.pl

We consider the classic Lagrange long gravitational wave of a homogeneous incompressible fluid in a shallow canal with a corrugated bottom. We use the asymptotic expansion method to find the effective depth of a one-dimensional canal and, hence, the effective wave velocity. A flow in a two-dimensional tank with a corrugated bottom is also studied by this method.

Keywords: bottom profiles, homogenization approach, effective depth

1. Introduction

The free surface of a fluid in equilibrium under the gravitational field is a plane. If due to an external perturbation the surface is moved from its equilibrium state, motion will arise in the fluid. This movement spreads in the fluid in the form of waves called gravitational, because their existence is conditioned by the field of gravity. Gravity waves occur mainly on the surface of the fluid, but they affect the interior also, the less the inner layers are deeper.

The restoration of the fluid to equilibrium will produce a movement of the fluid back and forth, called a wave orbit (Lamb, 1916; Landau and Lifshitz, 1987; Lighthill, 2001).

This type of waves include huge floating elevations of the sea water level (ocean tides) which, with regularity dictated by the lunar rotation, roll over the surface of water and sometimes even fall into river beds and run against their currents. In the Earth's atmosphere, gravity waves are a mechanism that produces the transfer of momentum from the troposphere to the stratosphere and mesosphere (Gill, 1982; Nappo, 2012). As for the flows of viscous fluids with the assumption of corrugated walls of the channel, they were studied thoroughly in the works of Mityushev and his school, e.g.: Adler *et al.* (2013), Malevich *et al.* (2006).

Introductions about long wave equations (shallow water) can be found in the monograph by Mei *et al.* (2005). The older book by Dingemans and the recent book by Popescu are providing a review of techniques available for the problems of wave propagation in regions with uneven beds as they are encountered in coastal areas (Dingemans, 1997; Popescu, 2014).

The article by Karaeva *et al.* (2018) deserves attention. The linearized system of shallow water equations was analyzed, and the homogenization method was used to solve the Cauchy problem for the wave equation describing evolution of the free surface elevation when long waves propagate in a basin over an uneven bottom. It was proved that, under certain conditions on the function describing the basin depth, the solution of the homogenized equation asymptotically approximates the solution of the exact equation. Several examples for calculating the model one-dimensional wave equation are constructed using real world ocean bathymetry data. We consider the case of gravity waves whose length is great compared to depth, which are called long waves. In description introduced by Joseph Louis Lagrange (1781), these waves can be considered approximately linear.

In a linear approximation, we primarily use the principle of mass conservation, i.e. the volume conservation in the case of incompressible liquids, and the Euler equation for the main direction of fluid flow.

In this study, we will deal with the propagation of long gravitational waves in corrugated bottom canals or tanks, i.e. the case when the liquid depth varies. This is an important case, dominating in nature, where most of the natural rivers and streams have a folded bottom covered with various irregularities, stones or other obstacles. In particular, it is a case arriving in lakes, streams, rivers and open channels, in which there grow different types of plants, cf. Evangelos (2012), Kubrak *et al.* (2013), Telega and Bielski (2003), Wojnar and Bielski (2014).

Plan of the paper: First, the essentials of asymptotic homogenization are recapitulated (Section 2). Next we bring out the equations of the linear long gravitational wave in the canal to accurately capture the influence of corrugation of the bottom. The equations of motion of long gravitational waves running in one-dimensional canals (Section 3) and two-dimensional tanks with an uneven bottom (Section 5) are derived. In Section 4. we deal with 1D flows in reservoirs or canals with different particular types of periodic unevenness. The particular cases of the bottom are considered in detail. We are looking for a substitute canal with an even bottom for the canal with an uneven bottom. We use the method of asymptotic expansions, also known as the homogenization method.

2. Preliminaries on homogenization

The method described in the books of Bensoussan *et al.* (1980), Sanchez-Palencia (1980), Bakhvalov and Panasenko (1989) was recently developed in the works of Andrianov *et al.* (2018). These authors applied homogenization methods not only for periodic but also quasi-periodic structures, and optimized designs of functionally graded materials. The homogenization problems with special stress to fluid flows are discussed in the monograph by Mei and Vernescu (2010).

We consider long gravity waves in a canal with a corrugated bottom (1D) or in a tank with a corrugated bottom (2D). To describe the corrugation we use a concept of waviness spacing. In the lateral direction along the surface, the waviness spacing is a parameter that describes the mean spacing between periodic corrugation peaks. The amplitude and period of these wavy variations are small, and we call it the micro-waviness.

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain (in our case the bottom of a canal or a tank) and $\Gamma = \partial\Omega$ its boundary. The value $n = 1$ for 1D case and $n = 2$ for 2D case. We introduce a parameter $\varepsilon = l/L_\Omega$, where l and L_Ω are typical length scales of the structural micro-waves in the bottom and of the region Ω, respectively.

Accordingly to the two-scale asymptotic approach, instead of one space variable x, we introduce two variables, macroscopic x and microscopic y, where $y = x/\varepsilon$, and instead of the function $f(x)$ consider the function $f(x, y)$. Now we consider the space $\Omega \times \square$ where the elementary cell $\square$ is a segment of the length Y in 1D case or a rectangle $Y_1 \times Y_2$ in 2D case. The variable x is defined in Ω and y in $\square$. For shortness, we write x for x_α and y for y_α with $\alpha = 1$ for 1D case and $\alpha = 1, 2$ for 2D case.

The fluid depth is of the form

$$h^\varepsilon = h(y_\alpha, t) \tag{2.1}$$

where

$$y_\alpha = \frac{x_\alpha}{\varepsilon} \quad \text{and} \quad \alpha = 1 \quad \text{for} \quad \text{1D} \qquad \text{or} \qquad \alpha = 1, 2 \quad \text{for} \quad \text{2D} \tag{2.2}$$

The coefficients and the fields of the problem are functions of ε, what indicates the superscript ε. Taking into account the formula for the total derivative (known as the chain rule) we have

$$\frac{\partial f(x,y)}{\partial x} \rightsquigarrow \frac{\partial f(x,y)}{\partial x} + \frac{1}{\varepsilon}\frac{\partial f(x,y)}{\partial y} \qquad \text{with} \qquad y = \frac{x}{\varepsilon}$$

where the superscript ε denotes the micro-periodicity of the relevant quantities.

Applying the method of two-scale asymptotic expansions, we write

$$H^{\varepsilon} = H^{\varepsilon}(x) = H^{(0)}(x,y) + \varepsilon^1 H^{(1)}(x,y) + \varepsilon^2 H^{(2)}(x,y) + \cdots$$

where the functions $H^{(i)}(x,y)$, $i = 0,1,2,\ldots$ are assumed to be Y-periodic. For shortness, we omit the argument t in the terms of this expansion. In detail, we should write obviously $H^{(i)} = H^{(i)}(x,y,t)$. It is tacitly assumed that all derivatives appearing in the asymptotic homogenization make sense.

3. One-dimensional gravity waves

The free surface of a fluid in equilibrium in a gravitational field is a horizontal plane. If, under the action of some external perturbation, the surface is moved from its equilibrium position at some point, motion will occur in the liquid. This motion will be propagated over the whole surface in the form of waves of the amplitude H_{ampl}, which are called gravity waves, since they are due to the action of the gravitational field. The fluid velocity has two components $\mathbf{v} = [v_1, v_2] \equiv [u, v]$ with u in the x_1-direction, and v in the x_2-direction.

We shall consider gravity waves in which the velocity of the moving fluid particles is so small that we may neglect the nonlinear term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in comparison with $\partial\mathbf{v}/\partial t$ in Euler's equation. The condition $(\mathbf{v}\cdot\nabla)\mathbf{v} \ll \partial\mathbf{v}/\partial t$ is equivalent to $H_{ampl} \ll L$, it means that the amplitude H_{ampl} of the oscillations $H = H(x_1,t)$ in the wave must be small compared with the wavelength L (Landau and Lifshitz, 1987).

Let us examine first the propagation of long waves in a rectangular canal with a corrugated (wavy) bottom. The canal is supposed to be of constant width b and of infinite length along the x_1-axis, see Fig. 1.

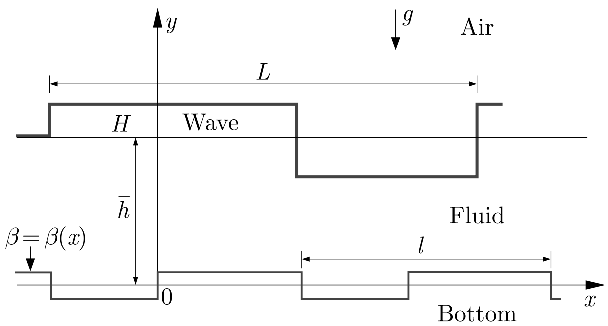

Fig. 1. Simplified idea of a one-dimensional gravity wave of height $H = H(x,t)$ moving past the canal with an uneven bottom resulting in different depths with the mean depth $\overline{h}$. The dimension of the elementary cell is l, the length of the wave L. The vector $\mathbf{g}$ denotes the Earth acceleration

The depth h of the fluid in the canal is supposed to be small in comparison with the length L of the considered fluid wave, $h \ll L$. The corrugation of the bottom is described by the periodic function $\beta = \beta(x_1)$. The mean value of β, is $\langle\beta(x_1)\rangle = 0$, and the mean depth of the fluid in the channel in equilibrium is $\overline{h}$.

The depth of the fluid in the canal in non-equilibrium at the point x_1 and at the time t is

$$h = h(x_1,t) = \overline{h} - \beta(x_1) + H(x_1,t) = h_0(x_1) + H(x_1,t)$$

where $h_0(x_1) \equiv \overline{h} - \beta(x_1)$ and $H = H(x_1,t)$ denotes the elevation of the wave surface above the equilibrium liquid free surface level.

The function $H = H(x_1,t)$ describing the profile of the considered gravity wave and the function $\beta = \beta(x_1)$ describing the bottom waviness are small in comparison with the mean fluid depth $\overline{h}$

$$H \ll \overline{h} \qquad \text{and} \qquad \beta \ll \overline{h}$$

The cross-section area of the liquid in the canal is given by

$$S(x_1,t) = bh(x_1,t)$$

We shall here consider longitudinal waves, in which the liquid moves along the canal. In such waves, the velocity component $v_1 \equiv u$ along the channel is large compared with the components $v_2 \equiv v$.

In an analogy to how the classic long wave equation is derived, but remembering that the depth $h_0(x_1)$ is variable, we get

$$\frac{\partial^2 H}{\partial t^2} - g\frac{\partial}{\partial x_1}\Big(h_0(x_1)\frac{\partial H}{\partial x_1}\Big) = 0 \tag{3.1}$$

If the bottom is flat $\beta = 0$ then $h_0 = \overline{h}$ and we receive

$$\frac{\partial^2 H}{\partial t^2} - g\overline{h}\frac{\partial^2 H}{\partial x_1^2} = 0 \tag{3.2}$$

This is the classical Lagrange *wave equation* (Lagrange, 1781), the value

$$c = \sqrt{g\overline{h}} \tag{3.3}$$

is the velocity of propagation of the long gravity wave with a small amplitude in the canal with the constant depth $\overline{h}$.

4. Homogenization of one-dimensional case

Notice, in this Section we write simply x instead of x_1.

Our equation of motion reads

$$\frac{\partial^2 H^\varepsilon}{\partial t^2} - g\frac{\partial}{\partial x}\Big(h_0^\varepsilon\frac{\partial H^\varepsilon}{\partial x}\Big) = 0 \tag{4.1}$$

or

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Big(H^{(0)}(x,y) + \varepsilon^1 H^{(1)}(x,y) + \varepsilon^2 H^{(2)}(x,y) + \cdots\Big) = g\Big(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\Big) \\ \cdot\Big[h_0(y)\Big(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\Big)\Big(H^{(0)}(x,y) + \varepsilon^1 H^{(1)}(x,y) + \varepsilon^2 H^{(2)}(x,y) + \cdots\Big)\Big]\end{aligned} \tag{4.2}$$

The argument t is omitted. According to the method of asymptotic homogenization, we compare the terms associated with the same power of ε. We successively obtain:

— At ε^{-2}

$$0 = \frac{\partial}{\partial y}\Big(h_0(y)\frac{\partial H^{(0)}(x,y)}{\partial y}\Big) \tag{4.3}$$

This equation is satisfied provided that $H^{(0)}$ does not depend on the local variable y, it is

$$H^{(0)} = H^{(0)}(x) \tag{4.4}$$

This statement holds true on the assumption that the coefficient $h_0(y)$ is Y-periodic. The result is easy to see if one multiplies Eq. (4.3) by $H^{(0)}$ and integrates over Y. After integration by parts, and since h_0 is positive for the integral to disappear, the derivative $\partial H^{(0)}(x,y)/\partial y$ must disappear.

— At ε^{-1} we receive

$$0 = \frac{\partial}{\partial y}\Big[h_0(y)\Big(\frac{\partial H^{(0)}(x)}{\partial x} + \frac{\partial H^{(1)}(x,y)}{\partial y}\Big)\Big] \tag{4.5}$$

and at ε^0

$$\begin{aligned}\frac{\partial^2 H^{(0)}(x)}{\partial t^2} &= g\frac{\partial}{\partial x}\Big[h_0(y)\Big(\frac{\partial H^{(0)}(x)}{\partial x} + \frac{\partial H^{(1)}(x,y)}{\partial y}\Big)\Big]\\ &+ g\frac{\partial}{\partial y}\Big[h_0(y)\Big(\frac{\partial H^{(1)}(x)}{\partial x} + \frac{\partial H^{(2)}(x,y)}{\partial y}\Big)\Big]\end{aligned} \tag{4.6}$$

Equation (4.5) is satisfied if

$$H^{(1)}(x,y) = \psi(y)\frac{\partial H^{(0)}(x)}{\partial x} \tag{4.7}$$

where $\psi(y)$ satisfies the equation

$$\frac{d}{dy}\Big[h_0(y)\Big(1 + \frac{d\psi(y)}{dy}\Big)\Big] = 0 \tag{4.8}$$

known as a *local problem*.

We submit expression (4.7) into ε^0-equation (4.6), integrate over Y and obtain

$$\frac{\partial^2 H^{(0)}(x)}{\partial t^2} = g\frac{\partial}{\partial x}\frac{1}{|Y|}\int_Y \Big[h_0(y)\Big(\frac{\partial H^{(0)}(x)}{\partial x} + \frac{\partial H^{(1)}(x,y)}{\partial y}\Big)\Big]\,dy \tag{4.9}$$

or

$$\frac{\partial^2 H^{(0)}(x)}{\partial t^2} = g h_0^{\mathit{eff}}\frac{\partial^2 H^{(0)}(x)}{\partial^2 x} \tag{4.10}$$

where

$$h_0^{\mathit{eff}} = \frac{1}{Y}\int_Y h_0(y)\Big(1 + \frac{d\psi(y)}{dy}\Big)\,dy \tag{4.11}$$

Equation (4.10) presents the homogenized equation of the long gravity wave in the channel with the effective depth h_0^{eff}. Evidently, by this equation we have also

$$c^{\mathit{eff}} = \sqrt{g h_0^{\mathit{eff}}} \tag{4.12}$$

the effective velocity of the gravity wave in the channel with a corrugated bottom, cf. Eq. (3.3).

Local equation (4.8) after the first integration gives

$$h_0(y)\Big(1+\frac{d\psi(y)}{dy}\Big)=C \tag{4.13}$$

where C is an unknown constant. After the second integration

$$\psi(y)-\psi(0)=\int\limits_0^y \Big(\frac{C}{h_0(y)}-1\Big)\,dy \tag{4.14}$$

and by the periodic boundary conditions on the elementary cell of the length Y

$$0=\psi(Y)-\psi(0)=\int\limits_0^Y \Big(\frac{C}{h_0(y)}-1\Big)\,dy \tag{4.15}$$

we get the constant

$$C=\frac{Y}{\int_0^Y \frac{dy}{h_0(y)}} \tag{4.16}$$

After substitution (4.13) into (4.11), we get

$$h_0^{\mathit{eff}}=\frac{1}{|Y|}C\int\limits_0^Y dy=C \tag{4.17}$$

what is our desired result.

Equations (4.17) and (4.16) can be also written in the form

$$h_0^{\mathit{eff}}=\frac{Y}{I_0} \tag{4.18}$$

where

$$I_0\equiv\int\limits_0^Y \frac{1}{h_0(y)}\,dy \tag{4.19}$$

Equation (4.10), like any one-dimensional wave equation, can have two solutions $f_1(x-c^{\mathit{eff}}t)$ and $f_2(x+c^{\mathit{eff}}t)$. The solution $f_1(x-c^{\mathit{eff}}t)$ represents what is called the *travelling plane wave* propagating in the positive direction of the x-axis. The solution $f_2(x+c^{\mathit{eff}}t)$ represents a wave propagating in the opposite direction.

4.1. The effective depth of the canal with a sinusoidal bed (Case I)

Consider a canal with the constant width w_0 and with the bed of a sinusoidal cross-section, with the elementary cell Y, see Fig. 2

$$h=h(y)=\overline{h}-a\cos(ky) \tag{4.20}$$

where

$$k=\frac{2\pi}{Y} \tag{4.21}$$

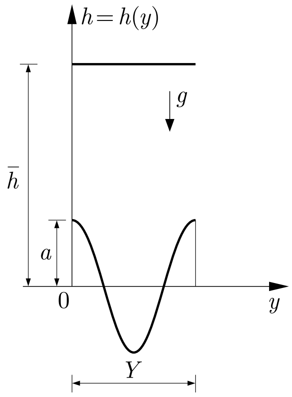

Fig. 2. Elementary cell Y of the canal with a sinusoidal profile of the bottom

In this case, by Eq. (4.19)

$$I_0 = \int\limits_0^Y \frac{dy}{\overline{h} - a\cos(ky)} = \frac{Y}{2\pi}\int\limits_0^{2\pi} \frac{dx}{\overline{h} - a\cos x} \equiv \frac{Y}{2\pi} I \tag{4.22}$$

We separately count the integral

$$I \equiv \int\limits_0^{2\pi} \frac{dx}{\overline{h} - a\cos x} = \int\limits_0^{\pi} \frac{dx}{\overline{h} - a\cos x} + \int\limits_{\pi}^{2\pi} \frac{dx}{\overline{h} - a\cos x} \tag{4.23}$$

After substitution $t = \tan(x/2)$, we have

$$\cos x = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad dx = \frac{2}{1+t^2}dt$$

and

$$I = \int\limits_0^{\infty} \frac{2\,dt}{\overline{h}(1+t^2) - a(1-t^2)} + \int\limits_{-\infty}^{0} \frac{2\,dt}{\overline{h}(1+t^2) - a(1-t^2)} = \frac{2\pi}{\sqrt{\overline{h}^2 - a^2}}$$

and by Eq. (4.18)

$$h_0^{eff} = \sqrt{\overline{h}^2 - a^2} \tag{4.24}$$

Since a is much less than $\overline{h}$, we can also write

$$h_0^{eff} = \overline{h}\Big(1 - \frac{1}{2}\frac{a^2}{\overline{h}^2}\Big) \tag{4.25}$$

4.2. The effective canal depths for other bed profiles (Cases II and III)

Consider canals with the constant width w_0 and with different bottom profiles.

Case II

The profile $h = h(y)$ of the cross-section of the cell Y is described by, see Fig. 3

$$h_0 = \begin{cases} \overline{h} - a & \text{for} \quad 0 < y < \frac{1}{4}Y \\ \overline{h} + a & \text{for} \quad \frac{1}{4}Y < y < \frac{3}{4}Y \\ \overline{h} - a & \text{for} \quad \frac{3}{4}Y < y < Y \end{cases} \tag{4.26}$$

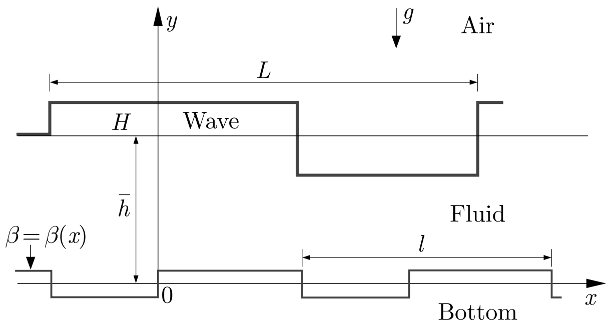

Fig. 3. Elementary cell Y for the channel with the step up step down profile of the bottom

By definition (4.19), and after integration

$$I_0 = \frac{\overline{h}}{\overline{h}^2 - a^2} Y \tag{4.27}$$

Hence, by Eq. (4.18)

$$h_0^{eff} = \frac{Y}{I_0} = \overline{h}\Big(1 - \frac{a^2}{\overline{h}^2}\Big) \tag{4.28}$$

Case III

The profile $h = h(y)$ of the cross-section of the cell Y is described by, see Fig. 4

$$h_0 = \begin{cases} \overline{h} - Ay - B & \text{for} \quad 0 < y < \frac{1}{2}Y \\ \overline{h} - A_1 y - B_1 & \text{for} \quad \frac{1}{2}Y < y < Y \end{cases}$$

with

$$A = -\frac{4a}{Y} \qquad B = a \qquad A_1 = \frac{4a}{Y} \qquad B_1 = -3a$$

By definition (4.19)

$$I_0 \equiv \int_0^Y \frac{dy}{h_0(y)} = \int_0^{Y/2} \frac{dy}{\overline{h} - Ay - B} + \int_{Y/2}^{Y} \frac{dy}{\overline{h} - A_1 y - B_1} = \frac{Y}{2a} \ln \frac{\overline{h} + a}{\overline{h} - a}$$

By Eq. (4.18)

$$h_0^{eff} = \frac{Y}{I_0} = \frac{2a}{\ln \frac{\overline{h}+a}{\overline{h}-a}} \tag{4.29}$$

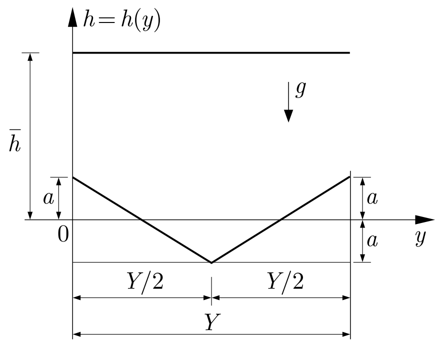

Fig. 4. Elementary cell Y for the canal with a saw-toothed profile of the bottom

For small a/h

$$h_0^{eff} = \overline{h}\Big(1 - \frac{1}{3}\frac{a^2}{\overline{h}^2}\Big) \tag{4.30}$$

Comparing the results of this Section, we have, cf. Eqs.(4.30), (4.25) and (4.28)

$$1 - \frac{1}{3}\frac{a^2}{\overline{h}^2} > 1 - \frac{1}{2}\frac{a^2}{\overline{h}^2} > 1 - \frac{a^2}{\overline{h}^2}$$

what means that for the same amplitude a the canal with the bed of the saw-toothed profile has the effective depth greater than other discussed profiles.

5. Long gravity waves in an infinite tank with the corrugated bottom

What concerns two-dimensional tanks with a corrugated bottom, due to the article length limit of 12 pages, in this Section we only provide a general scheme of proceeding according to the homogenization method. The equation of a long gravitational wave in two dimensions propagating into a tank of variable depth is

$$\frac{\partial^2 H}{\partial t^2} - g\frac{\partial}{\partial x_\alpha}\Big(h_0\frac{\partial H}{\partial x_\alpha}\Big) = 0 \qquad \alpha = 1,2 \tag{5.1}$$

If the bottom exhibits periodic micro-waviness, our equation reads

$$\frac{\partial^2 H^\varepsilon}{\partial t^2} - g\frac{\partial}{\partial x_\alpha}\Big(h_0^\varepsilon\frac{\partial H^\varepsilon}{\partial x_\alpha}\Big) = 0 \tag{5.2}$$

or

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Big(H^{(0)}(x,y) + \varepsilon^1 H^{(1)}(x,y) + \cdots\Big) &= g\Big(\frac{\partial}{\partial x_\alpha} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_\alpha}\Big) \\ \cdot\Big[h_0(y)\Big(\frac{\partial}{\partial x_\alpha} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_\alpha}\Big)&(H^{(0)}(x,y) + \varepsilon^1 H^{(1)}(x,y) + \cdots)\Big]\end{aligned} \tag{5.3}$$

According to the method of asymptotic homogenization we compare the terms associated with the same power of ε. We successively obtain:
— At ε^{-2}

$$0 = \frac{\partial}{\partial y_\alpha}\Big(h_0(y)\frac{\partial H^{(0)}(x,y)}{\partial y_\alpha}\Big) \tag{5.4}$$

This equation is satisfied provided that $H^{(0)}$ does not depend on the local variable y, it is

$$H^{(0)} = H^{(0)}(x) \tag{5.5}$$

This statement holds true on the assumption that the coefficient $h_0(y)$ is Y-periodic.
— At ε^{-1} we receive

$$0 = \frac{\partial}{\partial y_\alpha}\Big[h_0(y)\Big(\frac{\partial H^{(0)}(x)}{\partial x_\alpha} + \frac{\partial H^{(1)}(x,y)}{\partial y_\alpha}\Big)\Big] \tag{5.6}$$

— At ε^0

$$\begin{aligned}\frac{\partial^2 H^{(0)}(x)}{\partial t^2} &= g\frac{\partial}{\partial x_\alpha}\Big[h_0(y)\Big(\frac{\partial H^{(0)}(x)}{\partial x_\alpha} + \frac{\partial H^{(1)}(x,y)}{\partial y_\alpha}\Big)\Big]\\ &+ g\frac{\partial}{\partial y_\alpha}\Big[h_0(y)\Big(\frac{\partial H^{(1)}(x)}{\partial x_\alpha} + \frac{\partial H^{(2)}(x,y)}{\partial y_\alpha}\Big)\Big]\end{aligned} \tag{5.7}$$

Equation (5.6) is satisfied, if

$$H^{(1)} = \psi_\beta(y)\frac{\partial H^{(0)}(x)}{\partial x_\beta} \tag{5.8}$$

and the vector function $\psi_\alpha(y)$ is a solution of the following *local problem*

$$\frac{\partial}{\partial y_\alpha}\Big[h_0(y)\Big(\delta_{\alpha\beta} + \frac{\partial \psi_\beta(y)}{\partial y_\alpha}\Big)\Big] \tag{5.9}$$

Substituting (5.8) into (5.7) and integrating over Y, we obtain

$$\frac{\partial^2 H^{(0)}(x)}{\partial t^2} = g h^{\mathit{eff}}_{0\alpha\beta}\frac{\partial^2 H^{(0)}(x)}{\partial x_\alpha \partial x_\beta} \tag{5.10}$$

where

$$h^{\mathit{eff}}_{0\alpha\beta} = \frac{1}{Y}\int\limits_Y h_0(y)\Big(\delta_{\alpha\beta} + \frac{\partial \psi_\beta(y)}{\partial y_\alpha}\Big)\,dy \tag{5.11}$$

and the periodic boundary conditions over Y were exploited.

If $h_0(y_1,y_2) = h_0(y_2,y_1)$, a special case of which is the product equality

$$h_0(y_1,y_2) \equiv f(y_1)g(y_2) = f(y_2)g(y_1)$$

(f, g are arbitrary functions), then $\psi_1(y_1,y_2) = \psi_2(y_2,y_1)$ and $h^{\mathit{eff}}_{0\,11} = h^{\mathit{eff}}_{0\,22}$.

6. Comments

The method of two-scale asymptotic expansions, it would seem, only the MacLaurin expansion according to the ε coefficient (but in the functional space) made it possible to find effective depth equivalent values for long gravitational waves for various topographies of the bottom of the reservoir. Thanks to this, it is possible to describe the propagation of gravitational waves in a one-dimensional channel and a two-dimensional reservoir by the common wave equation with the velocity $\sqrt{gh^{\mathit{eff}}}$.

After deriving the equations of motion of long gravitational waves running in one-dimensional canals and two-dimensional tanks with an uneven bottom, we have proposed to apply the homogenization method estimating the influence of the rough bottom on the propagation of long gravity waves. We have shown that the equation describing such waves is the same as in the case of the flat bottom, but the depth of the reservoir is replaced by the effective value found according to the homogenization method.

If foreseeable wave disturbance is long, and if takes it for a length unit, then we have the inequalities $l \ll h \ll 1$, where l denotes the period of the bottom corrugation, and h – the depth of the canal liquid in equilibrium. The value of the bottom corrugation vertical amplitude a and the wave amplitude H on the surface of the fluid, that is its vertical disturbance H, both are of the order of l.

In our work, we dealt with the flow of long gravitational waves in one-dimensional canals and two-dimensional tanks with an uneven bottom. We assumed that the dimensions of unevenness were small in relation to the wavelength of the gravity wave. We were looking for a substitute reservoir (canal) with an even bottom for the realistic reservoir with an uneven bottom. The method of asymptotic development, known as the homogenization method, is well suited for this purpose.

Acknowledgments
The authors thank to unknown Reviewer for suggestions leading to the present form of the paper.

This work was partially supported within statutory activities No 3841/E-41/S/2020 of the Ministry of Science and Higher Education of Poland.

References

1. Adler P.M., Malevich A.E., Mityushev V.V., 2013, Nonlinear correction to Darcy's law for channels with wavy walls, *Acta Mechanica*, **224**, 8, 1823-1848
2. Andrianov I.V., Awrejcewicz J., Danishevskyy V.V., 2018, *Asymptotical Mechanics of Composites: Modelling Composites without FEM*, Springer International Publishing AG, Cham
3. Bakhvalov N.S., Panasenko G.P., 1989, *homogenization: Averaging Processes in Periodic Media: Mathematical Problems in the Mechanics of Composite Materials*, Kluwer, Dordrecht-Boston-London
4. Bensoussan A., Lions J.-L., Papanicolaou G., 1980, *Asymptotic Analysis of Periodic Structures*, North Holland Publishing Company, Amsterdam-New York-Oxford
5. Dingemans M.W., 1997, *Water Wave Propagation Over Uneven Bottoms, Part 1 – Basic Equations, Part 2 – Wave propagation formulation*, Advanced Series on Ocean Engineering: Vol. 13, World Scientific, Delft
6. Evangelos K., 2012, The impact of vegetation on the characteristics of the flow in an inclined open channel using the PIV method, *Water Resources and Ocean Science*, **1**, 1, 1-6
7. Gill A.E., 1982, *Gravity wave*, Atmosphere Ocean Dynamics, Academic Press
8. Karaeva D.A., Karaev A.D., Nazaikinskii V.E., 2018, Homogenization method in the problem of long wave propagation from a localized source in a basin over an uneven bottom, *Differential Equations*, **54**, 8, 1057-1072
9. Kubrak E., Kubrak J., Rowiński P.M., 2013, Application of one-dimensional model to calculate water velocity distributions over elastic elements simulating Canadian waterweed plants (Elodea canadensis), *Acta Geophysica*, **61**, 1, 194-210
10. Lagrange J.-L., 1781, Mémoire sur la théorie du mouvement des fluides, *Nouveaux mémoires de l'Académie royale des sciences et belles-lettres de Berlin*, année 1781, also: Oeúvres complétes, tome 4, 695-748

11. Lamb H., 1916, *Hydrodynamics*, 4th ed., University Press, Cambridge

12. Landau L.D., Lifshitz E.M., 1987, *Fluid Mechanics*, transl. by J.B. Sykes and W.H. Reid, 2nd ed., Pergamon Press, Oxford-New York

13. Lighthill J., 2001, *Waves in Fluids*, Cambridge University Press

14. Malevich A.E., Mityushev V.V., Adler P.M., 2006, Stokes flow through a channel with wavy walls, *Acta Mechanica*, **182**, 3-4, 151-182

15. Mei C.C., Stiassnie M., Yue D.K.-P., 2005, *Theory and Applications of Ocean Surface Waves*, Advanced Series on Ocean Engineering, World Scientific, Singapore, 1136 pp.

16. Mei C.C., Vernescu B., 2010, *Homogenization Methods for Multiscale Mechanics*, World Scientific Publishers, New Jersey-London-Singapore-Beijing-Shanghai-Hong Kong-Taipei-Chennai

17. Nappo C.J., 2012, *An Introduction to Atmospheric Gravity Waves*, 2nd ed. Waltham, Elsevier Academic Press, Massachusetts

18. Popescu I., 2014, *Computational Hydraulics – Numerical Methods and Modelling*, IWA Publishing Alliance House, London

19. Sanchez-Palencia E., 1980, *Non-homogeneous Media and Vibration Theory*, Springer Verlag, Berlin, Heidelberg, New York

20. Telega J.J., Bielski W., 2003, Flow in random porous media: effective models, *Computers and Geotechnics*, **30**, 4, 271-288

21. Wojnar R., Bielski W., 2014, Flow in the canal with plants on the bottom, [In:] *Complex Analysis and Potential Theory with Applications*, T. Aliev Azerogly, A. Golberg, S.V. Rogosin (Edit.), Cambridge Scientific Publishers, 167-183

Manuscript received November 25, 2020; accepted for print May 27, 2021