JOURNAL OF THEORETICAL AND APPLIED MECHANICS
52, 1, pp. 281-284, Warsaw 2014

SHORT RESEARCH COMMUNICATION

COUPLING MULTISTABLE SYSTEMS: UNCERTAINTY DUE TO THE INITIAL POSITIONS ON THE ATTRACTORS

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We consider the coupling of multistable nonidentical systems. For small values of the coupling coefficient the behavior of the coupled system strongly depends on the actual position of trajectories on their attractors in the moment when the coupling is introduced. After reaching the coupling threshold value, this dependence disappears. We give an evidence that this behavior is robust as it exists for a wide range of parameters and different types of coupling. We argue why this behavior cannot be considered as a dependence on the initial conditions.

Key words: coupled systems, synchronization, attractors

Multistability, e.g. the existence of several attractors for a given set of system parameters, is common in: weakly dissipative systems, systems involving a delay, and coupled systems (Feudel *et al.*, 1998). It has been observed in a large variety of systems in many areas of science (Feudel, 2008), namely, nonlinear optics, neuroscience, climate dynamics, laser physics, and electronic oscillators. Most studies performed so far dealt with systems which being uncoupled are monostable, with some exceptions (Pisarchik *et al.*, 2006).

We consider the dynamical system $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, p)$ or $d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, p)$ ($\mathbf{x} \in \mathbb{R}^n$, $p \in \mathbb{R}$ describes the system control parameter). Let for $p \in \mathbb{D} \subset \mathbb{R}$ the considered system be multistable. Assume that m of such systems (characterized by different values of p) become the subsystems of one coupled system. Let for $t = 0$ (i.e. at the moment when the coupling is introduced) each subsystem operates on a different co-existing attractor. For small values of the coupling coefficient ε, the behavior of the coupled system depends not only on the types of the attractors on which m subsystems operate but also on the actual positions of the subsystem trajectories on their attractors at the moment when the coupling is introduced. We call this type of behavior the *uncertainty due to the positions on the attractors*. Increasing the coupling strength after reaching the particular threshold value of the coupling coefficient ε, this dependence is no longer observed. This behavior is not simply the dependence on the initial conditions. Imagine that in the experimental case we have n multistable systems operating on different attractors. At one moment ($t = 0$), all n systems are coupled together creating one augmented system. As typically in the experiments, one cannot estimate the exact values of the systems state at $t = 0$ and the behavior for $\varepsilon > 0$ cannot be predicted. The behavior of coupled systems can be predicted and controlled only for a large coupling.

To illustrate our finding, we use examples: the coupled Hénon maps (discrete system) and coupled excited van der Pol-Duffing oscillators (continuous system). Firstly, we focus on the Hénon maps

$$\begin{aligned} x_{n+1}^{(i)} &= 1 - p_i {x_n^{(i)}}^2 + y_n^{(i)} + \varepsilon(y_n^{(i-1)} - y_n^{(i)}) \\ y_{n+1}^{(i)} &= -b x_n^{(i)} \end{aligned} \tag{1}$$

where $i = 1, 2, 3$, p_i and b are constant and ε is the coupling coefficient. For $p_i \in [1.480, 1.485]$ and $b = 0.138$, each map exhibits multistability (Casas and Rech, 2012; Martínez-Zérega and Pisarchik, 2012; Sausedo-Solorio and Pisarchik, 2011). In our numerical simulations, we consider the following values $p_1 = 1.4807$ (uncoupled map has period-2 and period-6 attractors), $p_2 = 1.4820$ (uncoupled map has period-4 and period-12 attractors) and $p_3 = 1.4847$ (uncoupled map has period-4 and chaotic attractors). At the moment when the coupling has been introduced, the Hénon maps evolve respectively on period-2 (x_1, y_1-map) and period-4 (x_2, y_2 and x_3, y_3-maps) shown in the boxes in Fig. 1. The initial position of the subsystem trajectories

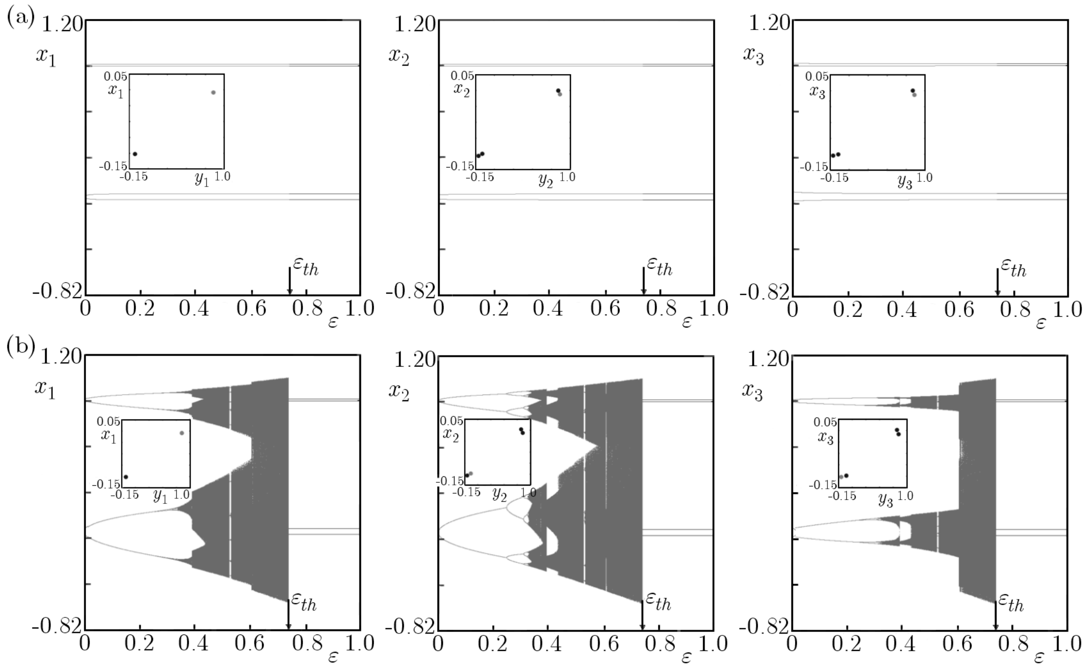

Fig. 1. Bifurcation diagrams of system (1): $b = 0.138$, $p_1 = 1.4807$, $p_2 = 1.4820$ and $p_3 = 1.4847$. Subsystems operates on different attractors as can be seen on the Poincaré maps shown in the small boxes

is indicated in light-grey color. In the example of Fig. 1a, the coupled system switches to the period-4 attractors for $\varepsilon \neq 0$ and stays on it then the whole considered interval of the coupling coefficient. In Fig. 1b, the bifurcation scenario is different. The evolution of the coupled system switches to the different period-4 orbit which with the increase of ε undergoes period doubling route to chaos. After reaching the threshold value ε_{th}, the chaotic attractor disappears and the system evolves on the same period-4 orbit as in the example of Fig. 1a. The same uncertainty effects have been observed for a larger number of the coupled Hénon maps (we consider up to 100 maps). The threshold value of the coupling coefficient ε_{th} seems to be independent of the number of maps.

As an example of the continuous system, we consider a ring of unidirectionally coupled externally exited van der Pol-Duffing oscillators

$$\ddot{x}_i - \alpha(1 - x_i^2)\dot{x}_i + x_i^3 = F \sin(p_i t) + \varepsilon(x_i - x_{i-1}) \tag{2}$$

where $i = 1, 2, 3$, α, p_i and F are constant. We assumed $\alpha = 0.2$, $F = 1$, $p_1 = 0.975$, $p_2 = 0.962$, $p_3 = 0.955$ and consider ε as a control parameter. In Chudzik *et al.* (2011), it was found that for $p_i \in (0.93, 0.98)$ each oscillator possessed plethora co-existing attractors of different types.

At the moment when the coupling is introduced $(t = 0)$ the subsystem trajectories have been respectively on 7, 18 and 31 periodic attractors shown in the boxes in Fig. 2. The actual position of the trajectories on the attractors is indicated in light-grey color. In Figs. 2a and 2b, we present the bifurcation diagrams of system (2) along the coupling coefficient ε. The results

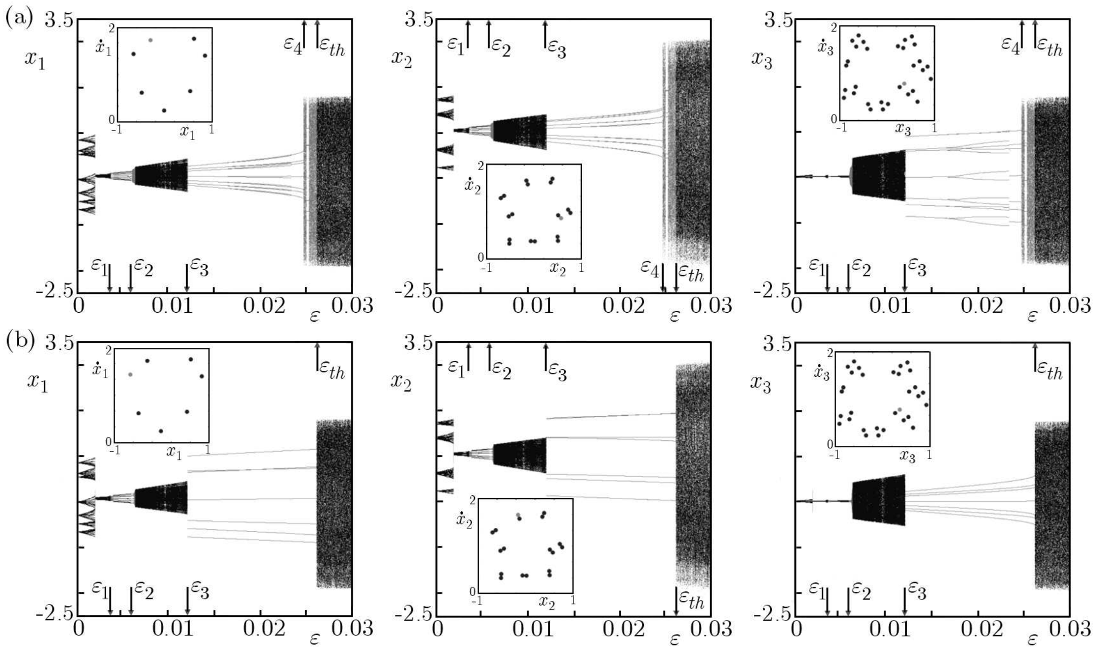

Fig. 2. Bifurcation diagrams of system (2): $\alpha = 0.2$, $F = 1$, $p_1 = 0.9750$, $p_2 = 0.9621$, $p_3 = 0.9552$. Subsystems operates on different attractors as indicated on the Poincaré maps shown in the small boxes

shown in Fig. 2a differ from those in Fig. 2b only by the subsystem trajectories positions on the attractors at $t = 0$. Comparing both bifurcations diagrams, one notices the differences (indicated in light-grey color). The dependence on the positions of the initial attractors is visible in small intervals around ε_1 and ε_2 and in the larger interval $[\varepsilon_3, \varepsilon_{th}]$. Notice that in the interval $[\varepsilon_4, \varepsilon_{th}]$, we observe the coexistence of different types of attractors. After the passage of the threshold value ε_{th}, the uncertainty disappears and the coupled systems evolve on the same chaotic attractor. The described uncertainty has been observed for a larger number of the coupled oscillators (we consider up to 100 oscillators) and also for different types of coupling (mutual, global). The threshold value of the coupling coefficient ε_{th} seems to be independent of the number of oscillators but differs for different types of coupling.

To summarize, we have shown that the coupling of multistable systems which operate on different attractors, we cannot predict the behavior of the system. For small coupling, the coupled system can operate on different coexisting attractors. The uncertainty due to the initial positions on the attractors seems to be common for the class of coupled systems with multistable subsystems. We observed it in a number of systems independently of the number of subsystems and type of coupling. This uncertainty may have practical implications. For some applications, the coexistence of attractors can be considered as undesired behavior. For example, for small values of the coupling coefficient $(\varepsilon < \varepsilon_{th})$, undesired behavior can be observed after the temporal breakdown of the coupling. After reestablishing of the coupling, one has to apply a special control procedure to reach the desired behavior again. From the point of view of the experimental systems, the described behavior cannot be simply considered as a dependence on the initial conditions.

Acknowledgment

This work has been supported by the Foundation for Polish Science, Team Programme – Project No. TEAM/2010/5/5.

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Manuscript received July 14, 2013; accepted for print October 7, 2013