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# TOPOLOGICAL CLASSES OF STATICALLY DETERMINATE BEAMS WITH ARBITRARY NUMBER OF SUPPORTS UNDER THE MOST UNFAVOURABLY DISTRIBUTED LOAD

Agata Kozikowska

*Bialystok University of Technology, Faculty of Architecture, Bialystok, Poland*

*e-mail: a.kozikowska@pb.edu.pl*

The paper deals with topological classes of statically determinate beams with an arbitrary number of pin supports. The beams carry piece-wisely distributed loads which are placed in such a way that bending moment values are extreme at any section. For such loads, it is sufficient to consider only two load cases with alternate spans uniformly loaded. Each beam with a fixed topology is subjected to geometrical optimization with the absolute maximum moment as the objective function. Exact formulas for optimal values of geometrical parameters are found for all topologies. An equality criterion between minimum values of the objective function is used as an equivalence relation. On the basis of this relation, the set of all topologies is divided into equivalence topological classes. Typical features of these classes are found and discussed.

*Key words:* statically determinate beams, topology and geometry optimization, equivalence classes; the worst load

## Notations

| | | |
|---|---|---|
| $c_E, c_H, c_S, c_T$ | – | number of external, internal cantilevers, number of segments $l_S$, $l_T$ |
| $g$ | – | number of optimal geometry variants |
| $\mathbf{h}, \mathbf{s}$ | – | coordinates of hinges and of supports |
| $l, l_E, l_H, l_S, l_T, L$ | – | lengths of optimal beam segments and length of beam, see Fig. 2 |
| $m$ | – | number of optimal envelopes of moment diagrams |
| $M_i, M_i^n$ | – | optimal moment value of topology $\mathbf{t}_i$ and class $\mathbf{T}_i^n$ |
| $n, p$ | – | number of supports and of topological classes, respectively |
| $q$ | – | maximum intensity of arbitrarily distributed gravitational load |
| $\{r_n\}$ | – | sequence of class moment ratios |
| $R$ | – | equivalence relation of beam topologies |
| $\mathbf{t}_i, \mathbf{t}_i^n$ | – | beam topology, $i = 1, 2, \ldots, \lvert\mathbf{T}^n\rvert$ or $i = 1, 2, \ldots, \lvert\mathbf{T}^{2:n}\rvert$ |
| $t_i$ | – | topological code of support $i$, $i = 1, 2, \ldots, n$ |
| $\mathbf{T}^n, \mathbf{T}^{2:n}$ | – | set of all topologies with $n$ supports and with two to $n$ supports |
| $\mathbf{T}_i^n, \mathbf{T}_i^{2:n}$ | – | topological class with $n$ supports and with two to $n$ supports |
| $\lvert\mathbf{T}^n\rvert, \lvert\mathbf{T}^{2:n}\rvert$ | – | number of topologies in set $\mathbf{T}^n$, $\mathbf{T}^{2:n}$ |
| $\{\mathbf{T}_k^n\}$ | – | sequence of topological classes |
| $x$ | – | axial coordinate |
| $y_i$ | – | dimensionless length of cantilever $i = 1, 2, \ldots, n$ |
| $z_i$ | – | dimensionless length of span $i = 1, 2, \ldots, n-1$ |
| $(\cdot)^n, (\cdot)^{2:n}, (\cdot)_i^n$ | – | quantities in set $\mathbf{T}^n$, $\mathbf{T}^{2:n}$ and class $\mathbf{T}_i^n$ |

## 1. Introduction

Beams are encountered all around us in many engineering applications. Statically determinate beams are also widely used in engineering structures due to their many advantages. Their short members are well suited for prefabrication, transportation and installation. In these beams, no stresses are produced by changes of temperature, settlement of supports and imperfections of assembly. Statically determinate beam cases are the basis of solid mechanics (Pedersen and Pedersen, 2009). They have not been fully explored yet and still attract attention of researchers (Golubiewski, 1995; Choi *et al.*, 2004; Pennock and Alwerdt, 2007; Liu *et al.*, 2009).

Topology optimization is a rapidly expanding research area of structural mechanics (Kirsch, 1989; Rozvany *et al.*, 1995; Bojczuk and Szteleblak, 2006). Topological optimization of beams is an important part of this area and can be found, among others, in articles of Mróz and Rozvany (1975), Imam and Al-Shihri (1996), Wang and Chen (1996), Bojczuk and Mróz (1998), Won and Park (1998), Mróz and Bojczuk (2003), Wang (2004, 2006), Friswell (2006), Jang *et al.* (2009). These papers, however, do not concern multispan hinged beams and the assignment of supports to bars. Topological optimization of statically determinate multispan hinged beams with arbitrary many supports was the subject of the author's earlier articles (Rychter and Kozikowska, 2009; Kozikowska, 2011). The topology in these papers is understood as the way how supports are connected to bars. The first paper introduces the space of these beams and a genetic algorithm for their topology and geometry optimization. The second presents topological classes of these beams. In both articles, the beams carry stationary loads which remain in a fixed position. In many practical situations, however, beams are subjected to a load whose position may vary. Such optimization tasks usually come down to problems with multiple load cases. There is a limited number of papers about beam optimization involving multiple loading conditions (Mayeda and Prager, 1967; Karihaloo and Kanagasundaram, 1988; Rozvany *et al.*, 1988; Bryant and Heinlein, 1994). Therefore, this article deals with topology and geometry optimization of statically determinate beams under the worst piece-wisely distributed load. It is assumed that the beams under consideration are loaded so slowly that the load may be regarded as quasi-static. In order to determine the most unfavourable arrangements of this load, influence lines are used. Since the beams primarily must resist bending due to action of transverse loads, the absolute maximum bending moment was chosen as the objective function to rank beam topologies, like in Wang (2006) and Xing and Wang (2012). According to influence lines, the maximum possible value of this function in each cross-section of the beam corresponds to only two quasi-static load cases with a distributed load, which covers all odd or all even spans.

Given the complexity of topology design spaces, topological optimization is usually not carried out in the full domain. Since the whole space of statically determinate beam topologies is known (Rychter and Kozikowska, 2009), exhaustive examination of all possibilities can be performed and a division of this space into topological equivalence classes can be found. This partition is based on the equivalence relation defined as an equality criterion between values of the absolute maximum moment of beams with optimal geometry. Geometry optimization of each beam with a fixed topology was carried out by a modified version of the genetic algorithm, which was presented in Rychter and Kozikowska (2009). Typical features of optimal geometries and exact formulas for optimal locations of supports and hinges are shown. A comparison of topological classes for a uniform load and the most unfavourably distributed load is reported.

## 2. Beam topology and geometry

In this paper, we study all statically determinate beams resting on a fixed number of pin supports or on a number of pin supports varying within a certain interval. Such beams were analysed in Rychter and Kozikowska (2009) and Kozikowska (2011). To find the topology of a beam, we start with the simply supported beam with all supports at the ends of the bars. Then we regard each support as topologically moveable: it can be moved inside its associated bar (first and last support) or inside any of its two associated bars (intermediate supports). The topology of a $n$-support statically determinate beam is a vector of these support shifts relative to the ends of bars: move left (code 1), move right (code 2), and no move (code 0)

$$\mathbf{t}_i^n = [t_1, \ldots, t_n] \tag{2.1}$$

The geometry of the beam is represented by a set of $n-1$ dimensionless lengths of spans between neighbouring supports (parameters $z_i$), two dimensionless lengths of external cantilevers (parameters $y_1$ and $y_n$) and $n-2$ ratios of internal cantilever lengths to span lengths (parameters $y_i$ for $i \in \{2, \ldots, n-1\}$). The size $|\mathbf{T}^n|$ of the set $\mathbf{T}^n$ of all $n$-support beam topologies and the size $|\mathbf{T}^{2:n}|$ of the set $\mathbf{T}^{2:n}$ of all topologies of beams with two to $n$ supports are given in Kozikowska (2011).

## 3. Equivalence relation of beam topologies

### 3.1. Geometry optimization of the beam with a fixed topology

The beams considered in Kozikowska (2011) were subjected to stationary loading. In this article, as in many practical situations, beams carry piece-wisely distributed loads that can occupy different positions. The maximum intensity of the arbitrarily distributed gravitational load is $q$. It is assumed that the rate of load changes is slow enough so that the load can be considered as a quasi-static. The effects of such loads are studied by means of influence lines in Fig. 1.

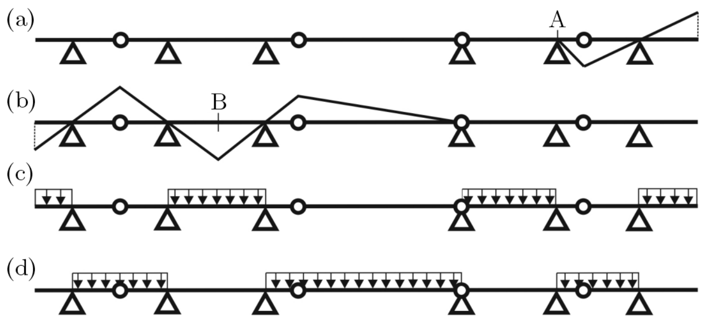


Fig. 1. A beam with the two most unfavourable cases of piece-wisely distributed load: (a) and (b) influence lines for the bending moment at the section A and B, (c) the load that causes the maximum possible value of bending moment with the top in tension in all sections of odd bars and with the bottom in tension in all sections of even bars, (d) the load that causes the maximum possible value of bending moment with the bottom in tension in all sections of odd bars and with the top in tension in all sections of even bars

We observe that the bending moment reaches the maximum value with the top or the bottom in tension when piece-wisely distributed loads occupy all the spans of the beam, over which the influence line does not change sign. Furthermore, the most unfavourable load out of loads of any distribution is uniform of maximum intensity $q$. The most dangerous loads can be extended to some spans of the beam where the ordinates of the influence line are equal to zero. This gives the two most unfavourable load cases with the load on all odd or all even spans, regardless of the topology of the beam and the place where the bending moment is calculated, as shown in Fig. 1. The first load case from Fig. 1c creates the largest possible moment values with the top in tension in all sections of odd bars and with the bottom in tension in all sections of even bars. The second load case from Fig. 1d produces the largest possible moment values on the opposite side of he beam: for odd bars tension appears at the bottom fibres and for even bars at the top fibres.

We consider a beam of unit length, with a fixed topology. The beam is optimized with respect to geometrical variables for both the most unfavourable load cases. This optimization problem may be formulated as follows

$$\begin{aligned}&\text{Minimize} \quad \max_{x\in[0,1]} |M(z_i, y_j, x)|\\&\text{Subject to} \quad \begin{cases} 0 < z_i < 1 \quad i = 1,2,\dots,n-1 \\ 0 < y_j < 1 \quad \text{for} \quad t_j \neq 0 \quad j = 1,2,\dots,n \\ y_1 + z_1 + z_2 + \ldots + z_{n-1} + y_n = 1 \end{cases}\end{aligned} \tag{3.1}$$

where $\max_{x\in[0,1]} |M(z_i, y_j, x)|$ is the objective function representing the maximum of the absolute bending moment for both load cases, $z_i$ denote lengths of spans, $y_j$ represent nonzero lengths of internal and external cantilevers with nonzero topological codes and $x$ is the axial coordinate. The total number of design variables is equal to the sum of the number of spans and the number of nonzero external and internal cantilever lengths.

Optimization of geometrical parameters for all topologies $\mathbf{t}_i$ for both load cases was made using a modified version of the genetic algorithm (Rychter and Kozikowska, 2009). Chromosomes representing a beam with a fixed topology are vectors of real genes $z_i$ and $y_i$. Such chromosomes are compact and suitable for genetic transformations, particularly crossover, which operation is hard to design for statically determinate beams. The minimal value of the absolute maximum bending moment $M_i$ is a result of geometry optimization of each beam with topology $\mathbf{t}_i$ for both load cases.

### 3.2. Definition of equivalence relation of beam topologies

$\mathbf{T}$ is the set of beam topologies: $\mathbf{T}^n$ or $\mathbf{T}^{2:n}$. We define an equivalence relation $R$ on the set $\mathbf{T}$. Any two topologies $\mathbf{t}_i$ and $\mathbf{t}_j$ of the set $\mathbf{T}$ are equivalent with respect to the relation $R$ if the values of the optimal moments $M_i$ and $M_j$ of these topologies are equal

$$\mathbf{t}_i \equiv_R \mathbf{t}_j \qquad \text{if} \quad M_i = M_j \tag{3.2}$$

The relation $R$ creates a decomposition of the set $\mathbf{T}^n$ into disjoint equivalence classes of beam topologies or topological classes $\mathbf{T}_i^n$ and the set $\mathbf{T}^{2:n}$ into topological classes $\mathbf{T}_i^{2:n}$, respectively. Parameters from the class $\mathbf{T}_i^n$ have the superscript $n$ and subscript $i$.

## 4. Topological classes for a fixed number of supports

### 4.1. Optimal envelope of bending moment diagrams for a fixed topology

A beam of length $L$ from the class $\mathbf{T}_i^n$, with optimal geometry for a fixed topology, is shown in Fig. 2. The beam is found with two unique bending moment diagrams, drawn with a solid line or dashed line, for both the most unfavourable load cases. The optimal envelope of the two moment diagrams has the same local extreme moment values equal to $M_i^n$. These values occur at the bottom of the beam at mid spans or close to them, over all supports which were moved away from the ends of bars and over whole spans which were created by moving away both supports from the ends of bars towards the middle of the bars. Both moment diagrams are never on the same side of the beam. The envelope may have portions with non-zero values only on one side of the beam. Such portions are cantilevers at the end of the beam and spans with two zero moment points inside or at the end (such as hinges and/or ends of the beam). Zero values of the envelope occur only in hinges and at both ends of the beam. Unsupported hinges can not change their locations without changing the moment diagrams like in the case of stationary loading (Kozikowska, 2011) and the optimal envelope of moment diagrams is equivalent to only one topology.

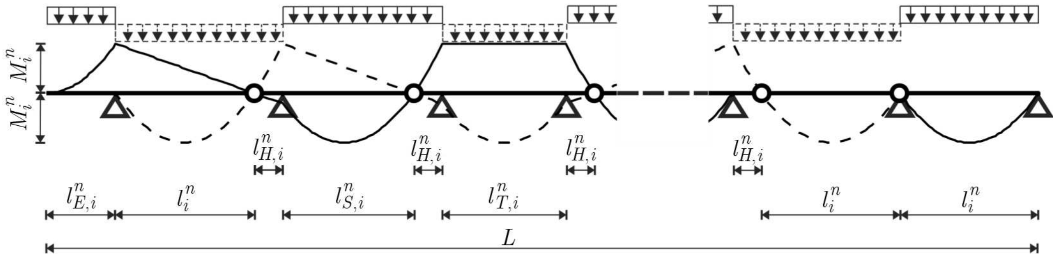


Fig. 2. A beam with optimal geometry for a fixed topology from the class $\mathbf{T}_i^n$

The paper provides exact formulas that allow one to calculate the optimal geometry for any topology, under the most unfavourably distributed load. To get the values of the parameters $l_i^n$, $l_{E,i}^n$, $l_{H,i}^n$, $l_{S,i}^n$ and $l_{T,i}^n$ (see Fig. 2) we need to solve the system of equations given below

$$
\begin{aligned}
&(n - c_{S,i}^n - c_{T,i}^n - 1)l_i^n + c_{E,i}^n l_{E,i}^n + c_{H,i}^n l_{H,i}^n + c_{S,i}^n l_{S,i}^n + c_{T,i}^n l_{T,i}^n = L \\
&\frac{1}{2}l_i^n - l_{E,i}^n = 0 \qquad\qquad (l_i^n)^2 - 4l_i^n l_{H,i}^n - 4(l_{H,i}^n)^2 = 0 \\
&l_{S,i}^n = \frac{1}{2}l_i^n + \frac{1}{2}\sqrt{(l_i^n)^2 - l_i^n l_{H,i}^n} \qquad\qquad l_{T,i}^n = \sqrt{(l_i^n)^2 - l_i^n l_{H,i}^n}
\end{aligned}
\tag{4.1}
$$

where $l_{E,i}^n$ and $l_{H,i}^n$ denote the lengths of nonzero external and internal cantilevers, respectively, $l_i^n$, $l_{S,i}^n$ and $l_{T,i}^n$ are the lengths of optimal beam segments between two closest supports and/or hinges with a parabolic moment diagram at the bottom of the beam for one of the two load cases. Zero moment values for this load case occur at both ends of the segment $l_i^n$, at only one end of the segment $l_{S,i}^n$ and do not occur at any point of the segment $l_{T,i}^n$. The first equation in (4.1) describes the total length of the beam. The parameters $c_{E,i}^n$, $c_{H,i}^n$, $c_{S,i}^n$, $c_{T,i}^n$ are the numbers of the segments $l_{E,i}^n$, $l_{H,i}^n$, $l_{S,i}^n$ and $l_{T,i}^n$, respectively. The second equation represents the comparison between the lengths of a cantilever and a simply supported beam with the same values of the absolute maximum moment. The maximum bending moment value of the simply supported beam of the length $l_i^n + 2l_{H,i}^n$ equals twice this value of the simply supported beam of the length $l_i^n$ in accordance with the third equation. The forth and the last equations in (4.1) are computed from the system of equations (4.2) and (4.3) and are explained graphically in Fig. 3 and Fig. 4, respectively

$$
l_{S,i}^n = l_i^n - l_X \qquad \frac{l_{H,i}^n}{M_X} = \frac{l_i^n}{M_i^n} \qquad M_X = \frac{1}{2}q l_i^n l_X - \frac{1}{2}q(l_X)^2 \qquad M_i^n = \frac{1}{8}q(l_i^n)^2 \tag{4.2}
$$

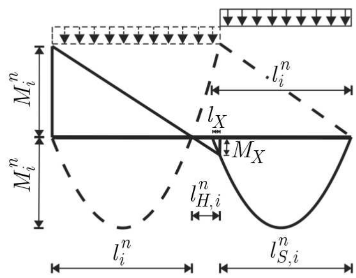


Fig. 3. Graphic explanation of equations (4.2)

The first equation in (4.2) specifies the difference of the segment lengths $l_i^n$ and $l_{S,i}^n$ (see Fig. 3). The second equation is obtained from the similarity of the right angle triangles with the bases $l^n_{H,i}$ and $l^n_i$. The bending moment values $M_X$ and $M^n_i$ are calculated for the simply supported, uniformly loaded beam of the length $l^n_i$: $M_X$ at the distance $l_X$ from the end of this beam and $M^n_i$ in the middle of this beam (the third and the last equation)

$$l^n_{T,i} = l^n_i - 2l_X \qquad \frac{l^n_{H,i}}{M_X} = \frac{l^n_i}{M^n_i} \qquad M_X = \frac{1}{2}ql^n_i l_X - \frac{1}{2}q(l_X)^2 \qquad M^n_i = \frac{1}{8}q(l^n_i)^2 \tag{4.3}$$

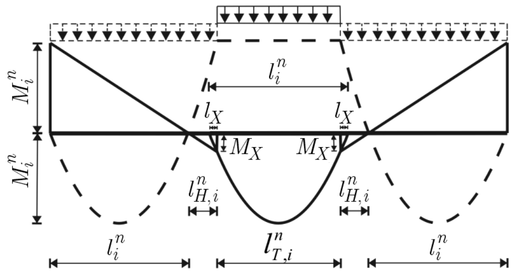


Fig. 4. Graphic explanation of equations (4.3)

The first equation in (4.3) determines the length of the segment $l^n_{T,i}$ (see Fig. 4). The other equations in (4.3) are the same as in (4.2).

The solution to system (4.1) is

$$\begin{aligned} &l^n_i = \frac{L}{d^n_i} \qquad l^n_{E,i} = \frac{L}{2d^n_i} \qquad l^n_{H,i} = (\sqrt{2}-1)\frac{L}{2d^n_i} \\ &l^n_{S,i} = \left(\sqrt{\tfrac{3}{2} - \tfrac{\sqrt{2}}{2}} + 1\right)\frac{L}{2d^n_i} \qquad l^n_{T,i} = \sqrt{\tfrac{3}{2} - \tfrac{\sqrt{2}}{2}}\,\frac{L}{d^n_i} \end{aligned} \tag{4.4}$$

where

$$d^n_i = n + \frac{1}{2}\Big(c^n_{E,i} + (\sqrt{2}-1)c^n_{H,i}\Big) + \left(\sqrt{\frac{3}{2} - \frac{\sqrt{2}}{2}} - 1\right)\Big(\frac{1}{2}c^n_{S,i} + c^n_{T,i}\Big) - 1$$

The lengths of two-hinged spans of the optimal beam are equal to $l^n_i + 2l^n_{H,i}$. One-hinged spans have the lengths of $l^n_i + l^n_{H,i}$ or $l^n_{S,i} + l^n_{H,i}$. The lengths of spans without hinges are equal to $l^n_i$ or $l^n_{S,i}$ or $l^n_{T,i}$. The values of the parameters $c^n_{E,i}$, $c^n_{H,i}$, $c^n_{S,i}$, $c^n_{T,i}$ can be calculated from the beam topology $\mathbf{t}^n_i$ (see Eq. (2.1)). The value of the parameter $c^n_{E,i}$ is equal to the number of non-zero elements in the first and last position of the code $\mathbf{t}^n_i$. The value of the parameter $c^n_{H,i}$ is the number of non-zero elements in positions 2 through $n-1$ of the code $\mathbf{t}^n_i$. The value of the parameter $c^n_{S,i}$ equals the number of pairs of neighbouring code elements equal to 11 or 22 (where for example 111 denotes two pairs) excluding pairs in the sequence 2211. The parameter $c^n_{T,i}$ can be calculated as the number of the sequences 2211 in the topological code $\mathbf{t}^n_i$. Algorithms to calculate the coordinates of supports and hinges (on the basis of the beam topology and the lengths $l^n_i$, $l^n_{E,i}$, $l^n_{H,i}$, $l^n_{S,i}$, $l^n_{T,i}$ and $L$) are given by a pseudo code in Appendix.

### 4.2. Features of beam topologies in a topological class

All optimal bending moment diagram pairs from a topological class, under the most dangerous piece-wisely distributed load, are shown in Fig. 5.

All topologies in the topological class $\mathbf{T}^n_i$ have the same value of moment $M^n_i$, which is dependent on the length $l^n_i$, in accordance with Eqs. (4.2) and (4.3). According to Eq. (4.4), the length $l^n_i$ depends on the number of supports and the values of the parameters $c^n_{E,i}$, $c^n_{H,i}$,

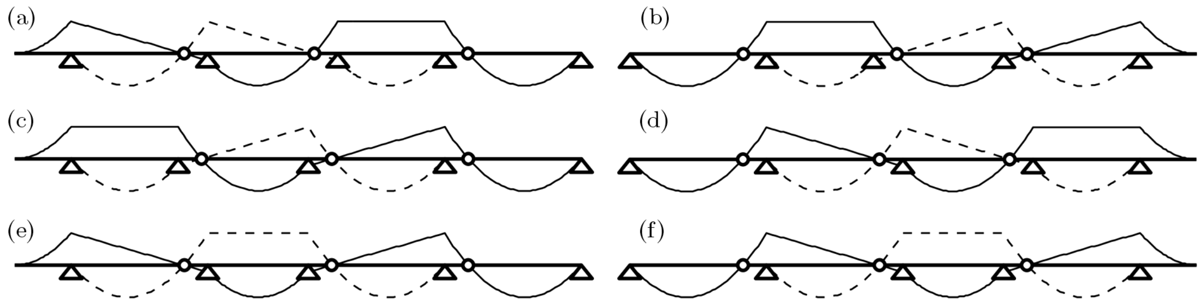


Fig. 5. All optimal envelopes of moment diagrams in the class $\mathbf{T}^5_{10}$: (a)-(d) $c^5_{E,10} = 1$, $c^5_{H,10} = 3$, $c^5_{S,10} = 2$, $c^5_{T,10} = 0$, (e)-(f) $c^5_{E,10} = 1$, $c^5_{H,10} = 3$, $c^5_{S,10} = 0$, $c^5_{T,10} = 1$

$c^n_{S,i}$ and $c^n_{T,i}$. In addition, the parameter $c^n_{S,i} = 2$ is equivalent to the parameter $c^n_{T,i} = 1$ (see Fig. 5). Thus for two topologies $\mathbf{t}_i$ and $\mathbf{t}_j$ of the set $\mathbf{T}^n$ under the most dangerous piece-wisely distributed load, the equivalent condition from Eq. (3.2) can be expressed as

$$\mathbf{t}_i \equiv_R \mathbf{t}_j \qquad \text{if} \qquad c_{E,i} = c_{E,j} \quad \wedge \quad c_{H,i} = c_{H,j} \quad \wedge \quad c_{S,i} + 2c_{T,i} = c_{S,j} + 2c_{T,j} \tag{4.5}$$

where $c_{E,i}$, $c_{H,i}$, $c_{S,i}$, $c_{T,i}$, $c_{E,j}$, $c_{H,j}$, $c_{S,j}$, $c_{T,j}$ are the numbers of appropriate segments for the topology $\mathbf{t}_i$ and $\mathbf{t}_j$, respectively.

### 4.3. Comparison of topological classes

The whole set of three-support topological classes, under the most unfavourably distributed load, with all optimal envelopes of moment diagrams is presented in Fig. 6. The better topological classes are, the more cantilevers and the less one-hinged spans their beams have. In other words, better classes have larger values of the parameters $c^n_{E,i}$ and $c^n_{H,i}$ and smaller values of the parameters $c^n_{S,i}$ and $c^n_{T,i}$. The values of the parameters $c^n_{E,i}$, $c^n_{H,i}$, $c^n_{S,i}$ and $c^n_{T,i}$ for the classes from Fig. 6 are given in Table 1. The best topologies in the first class $\mathbf{T}^n_1$ with an odd number of supports have only one one-hinged span (see Fig. 6a and Fig. 7a). The number of topologies in the best classes with the odd number of supports is equal to $n-1$. The best topological classes with an even number of supports have only one topology, which does not have any one-hinged spans (see Fig. 7b).

**Table 1.** Values of the parameters $c^n_{E,i}$, $c^n_{H,i}$, $c^n_{S,i}$ and $c^n_{T,i}$ in tree-support topological classes

| $\mathbf{T}^3_i$ | $\mathbf{T}^3_1$ | $\mathbf{T}^3_2$ | $\mathbf{T}^3_3$ | $\mathbf{T}^3_4$ | $\mathbf{T}^3_5$ | $\mathbf{T}^3_6$ | $\mathbf{T}^3_7$ |
|---|---|---|---|---|---|---|---|
| $c^n_{E,i}$ | 2 | 2 | 1 | 1 | 1 | 0 | 0 |
| $c^n_{H,i}$ | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| $c^n_{S,i}$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| $c^n_{T,i}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Information about the ways of hinge placement in statically determinate multispan beams can be found in many books on structural mechanics (Darkov and Kuznetsov, 1970; Nowacki, 1976; Kolendowicz, 1993; Hartsuijker and Welleman, 2006; Gambhir, 2009, 2011; Karnovsky and Lebed, 2010). The authors distribute hinges at spans and/or on supports of continuous beams and then they study properties of generated statically determinate beams. They state that by choosing an adequate place for the hinges, it is possible to influence the absolute maximum bending moment positively. They notice that this moment in a multispan beam with hinges away from supports is smaller than in a series of simple beams with the same spans (when hinges are on supports). Moreover, some authors recommend the hinge layout with the smallest possible

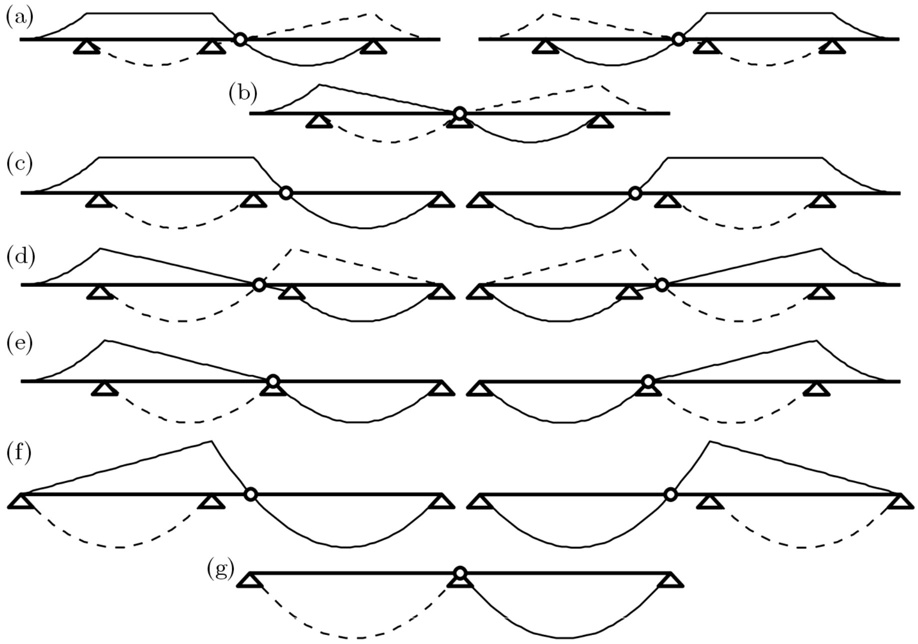


Fig. 6. All three-support topological classes with their optimal envelopes of moment diagrams: (a) $\mathbf{T}_1^3$, (b) $\mathbf{T}_2^3$, (c) $\mathbf{T}_3^3$, (d) $\mathbf{T}_4^3$, (e) $\mathbf{T}_5^3$, (f) $\mathbf{T}_6^3$, (g) $\mathbf{T}_7^3$

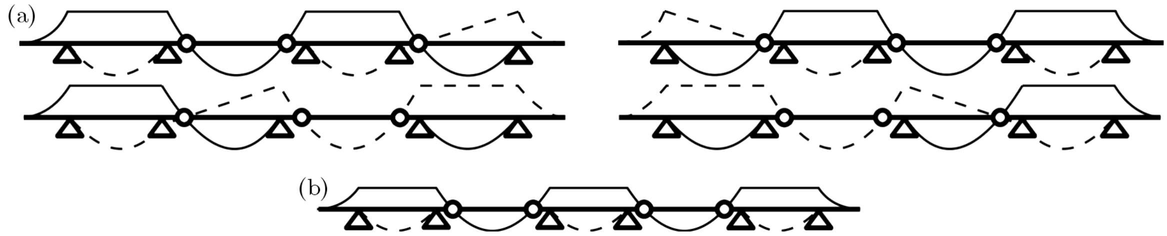


Fig. 7. The first five- and six-support topological classes: (a) $\mathbf{T}_1^5$, (b) $\mathbf{T}_1^6$

number of one-support bars (Nowacki, 1976; Gambhir, 2009, 2011). This layout is optimal for the absolute maximum bending moment under most unfavourably distributed load, but the authors do not justify their recommendation in this way. They advise to make the number of one-hinged spans as small as possible because the failure of any part of such a beam does not cause the entire beam to collapse.

An algorithm which calculates the number of topological classes for any number of support was written by the author. The sequence of numbers of classes for a growing number of supports was generated by this algorithm. The closed form of the formula for the total number of classes $p^n$ was found on the basis of this sequence using Mathematica software package

$$p^n = \frac{1}{4}[6n^2 - 12n + 11 + (-1)^n] \tag{4.6}$$

Let us consider the sequence of real numbers $\{r_n\}_{n=2}^{\infty}$ whose members are ratios of moment values of extreme classes. Calculating for both classes $l_i^n$ from Eq. (4.4) and $M_i^n$ from Eq. (4.3) and dividing one moment value by the other, we obtain

$$r_n = \frac{M^n_{\text{the last class}}}{M^n_{\text{the first class}}} = \frac{1}{(n-1)^2}\Big[n+\frac{1}{2}(\sqrt{2}-1)(n-2)+\frac{1}{2}\Big(\sqrt{\frac{3}{2}-\frac{1}{2}\sqrt{2}}-1\Big)(n \bmod 2)\Big]^2 \tag{4.7}$$

The sequence $\{r_n\}$ converges to the limit $(3+2\sqrt{2})/4$, which is smaller than the limit for stationary uniform loading equal to 2 (Kozikowska, 2011). The moment value of the last topological class without any cantilevers is the same for both the most unfavourably distributed load and uniform load. The moment value of the first class is obviously greater for the most unfavourably distributed load, which was selected from a variety of loads, including uniform over the entire beam. Therefore, elements of the sequence corresponding to the same value of $n$ and the limit of the sentence are smaller for the most unfavourable load. The rapprochement between class moment values with the growing number of supports is also seen in Fig. 8, which compares moment values $M_i^n$ of all topological classes for beams with 2, 3, 4 and 5 supports.

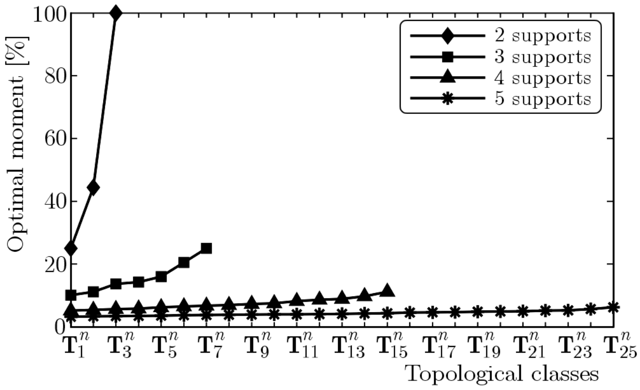


Fig. 8. Optimal moments in topological classes for a fixed number of supports

The number of different optimal envelopes of moment diagrams $m^n$ and the number of optimal geometry variants $g^n$ in all $n$-support topological classes are equal to the number of topologies $|\mathbf{T}^n|$, because an optimal envelope corresponds to only one topology and only one optimal geometry

$$m^n = g^n = |\mathbf{T}^n| = 4 \cdot 3^{n-2} \tag{4.8}$$

The formula for $|\mathbf{T}^n|$ is presented in Rychter and Kozikowska (2009).

## 5. Comparison of topological classes for a fixed number of supports under the most unfavourably distributed load und stationary uniform load

Statically determinate beams which are optimal for the absolute maximum moment have uniformly distributed bending moment diagrams for the stationary load (Kozikowska, 2011), whereas these beams have evenly distributed moment diagram envelopes for the worst piece-wisely distributed load. Topological classes are usually different for the stationary load than for the worst distributed load. Two topologies are members of the same class $\mathbf{T}_i^n$ under the stationary load if they have equal values of the parameters $c_{E,i}^n$ and $c_{H,i}^n$. For the most unfavourable load in addition to these parameters, the values of the parameters $c_{S,i}^n$ and $c_{T,i}^n$ also decide on the membership of topologies in the class $\mathbf{T}_i^n$. The first two- and three-support topological classes are the same in terms of topology for both types of loading. But optimal locations of supports and hinges are different. If $n > 3$ then the first class for the most unfavourable load with all supports shifted away from the ends of bars and with the minimal number of one-hinged spans (one for odd $n$ and zero for even $n$) is a part of the first class for the stationary loading. But optimal beams from the first class under the stationary load have all spans of equal lengths, while span lengths under the most unfavourable load are diverse. Two-hinged spans are the longest, one-hinged spans are shorter and the lengths of spans without hinges are the smallest. The optimal lengths of external and internal cantilevers are shorter under the stationary loading. The last topological classes with all supports under the ends of bars and with equal lengths of spans for these two loads are identical both in terms of topology and optimal geometry.

## 6. Topological classes for a different number of supports

Let us consider the set $\mathbf{T}^{2:n}$ of beam topologies with two to $n$ supports and topological classes $\mathbf{T}_i^{2:n}$. The plot in Fig. 9 shows the class moment values $M_i^{2:4}$ for beams with two up to four supports under the most unfavourably distributed load. The classes $\mathbf{T}_{16}^{2:4}$ and $\mathbf{T}_{21}^{2:4}$ contain topologies with two successive numbers of supports. The remaining classes consist of topologies with only one number of support.

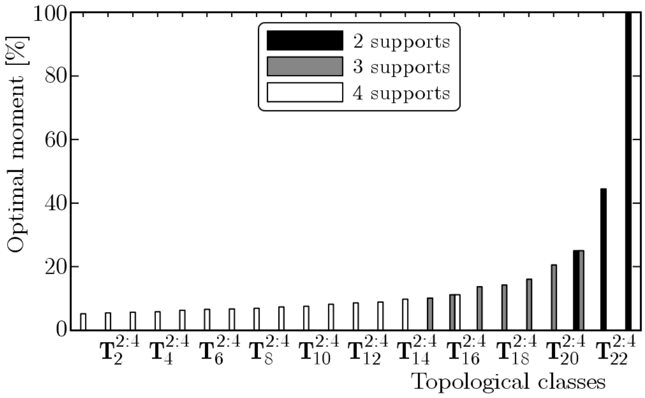


Fig. 9. Optimal moments in topological classes for a different number of supports

The beams from Fig. 10 belong to the same topological class $\mathbf{T}_i^{2:n}$ with two up to $n \geqslant 7$ supports. For a fixed number of supports, the six-support beams from Fig. 10a are elements of the topological class $\mathbf{T}_2^6$, the seven-support beams from Fig. 10b are members of the topological class $\mathbf{T}_{44}^7$.

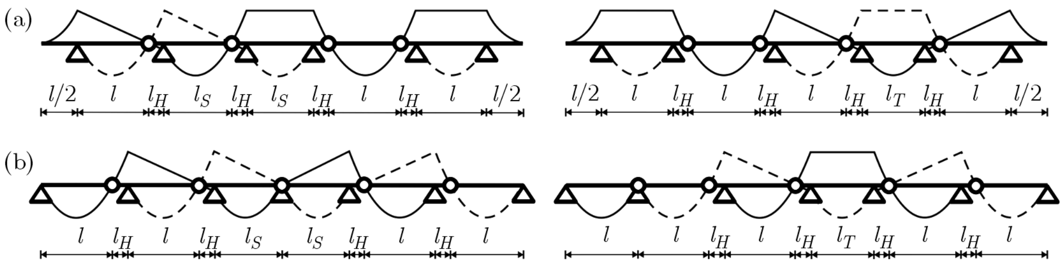


Fig. 10. Optimal beams with a different number of supports and equal values of the absolute maximum moment: (a) two topologies from the class $\mathbf{T}_2^6$, (b) two topologies from the class $\mathbf{T}_{44}^7$

The two topologies $\mathbf{t}_i^k$ and $\mathbf{t}_j^{k+1}$ with the number of supports $k$ and $k+1$, where $k \in \{2,\ldots,n-1\}$, are members of the same class $\mathbf{T}_i^{2:n}$ if they satisfy the condition

$$\mathbf{t}_i^k \equiv_R \mathbf{t}_j^{k+1} \qquad \text{if} \quad c_{E,i}=2 \ \wedge \ c_{E,j}=0 \ \wedge \ c_{H,i}=c_{H,j} \ \wedge \ c_{S,i}+2c_{T,i}=c_{S,j}+2c_{T,j} \tag{6.1}$$

where $c_{E,i}$, $c_{H,i}$, $c_{S,i}$, $c_{T,i}$, $c_{E,j}$, $c_{H,j}$, $c_{S,j}$, $c_{T,j}$ are the numbers of the segments $l_E$, $l_H$, $l_S$, $l_T$ for the topology $\mathbf{t}_i^k$ and $\mathbf{t}_j^{k+1}$, respectively.

## 7. Conclusions

The paper presents topology and geometry optimization of statically determinate beams. The beams can have any fixed number of supports or a number of supports from a certain interval. The objective of the optimization is to minimize the absolute maximum bending moment due to the most unfavourably distributed load. The maximum possible moment with the top and the bottom in tension occurs at any cross-section of a beam when alternate spans are uniformly loaded. A genetic algorithm is used to optimize the geometry of any beam with a fixed topology for two load cases. An optimal envelope of two moment diagrams has equal local extreme values for each beam. Algebraic formulas that determine values of optimal geometrical parameters are obtained by analyzing properties of the envelope. Beam topologies are sorted into topological classes according to minimal values of the objective function. The characteristic features of these classes are described and compared with those for stationary loading.

The results obtained here provide valuable guidance for the design of beam structures under the worst distributed load. Topological classes for the worst combination of distributed loads with fixed and the most unfavourable positions may be an important area of further research.

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## Appendix

Pseudocode for the algorithms to calculate the coordinates of supports and hinges of the optimal beam in a one-dimensional coordinate system with the origin at the left end of the beam

FUNCTION calculating_support_coordinates()
INPUT: the topological code of beam: $n$-element sequence $\mathbf{t}$; the lengths $l$, $l_E$, $l_H$, $l_S$, $l_T$, $L$
OUTPUT: n coordinates of supports: $n$-element vector $\mathbf{s}$
IF $t_1$ is equal to 0 THEN ASSIGN $s_1$ the value zero
ELSE ASSIGN s1 the value $l_E$
END IF
FOR $i$ starts at 2, $i < n$, increment $i$ DO
    IF $t_{i-1}$ is equal to 0 THEN
        IF $t_i$ is equal to 0 THEN ASSIGN $s_i$ the value $s_{i-1} + l$
        ELSE IF $t_i$ is equal to 1 THEN
            IF $t_{i+1}$ is equal to 1 THEN ASSIGN $s_i$ the value $s_{i-1} + l_S$
            ELSE ASSIGN $s_i$ the value $s_{i-1} + l$
            END IF
        ELSE ASSIGN $s_i$ the value $s_{i-1} + l + l_H$
        END IF
    ELSE IF $t_{i-1}$ is equal to 1 THEN
        IF $t_i$ is equal to 0 THEN ASSIGN $s_i$ the value $s_{i-1} + l + l_H$
        ELSE IF $t_i$ is equal to 1 THEN
            IF $t_{i+1}$ is not equal to 1 THEN ASSIGN $s_i$ the value $s_{i-1} + l + l_H$
            ELSE ASSIGN $s_i$ the value $s_{i-1} + l_S + l_H$
            END IF
        ELSE ASSIGN $s_i$ the value $s_{i-1} + l + 2l_H$
        END IF
    ELSE
        IF $t_i$ is equal to 0 THEN
            IF ($i$ is greater than 2) and ($t_{i-2}$ is equal to 2) THEN ASSIGN $s_i$ the value $s_{i-1} + l_S$
            ELSE ASSIGN $s_i$ the value $s_{i-1} + l$
            END IF
        ELSE IF $t_i$ is equal to 1 THEN
            IF ($i$ is equal to 2) or (($i$ is greater than 2) and ($t_{i-2}$ is not equal to 2)) THEN
                IF $t_{i+1}$ is not equal to 1 THEN ASSIGN $s_i$ the value $d_{i-1} + l$
                ELSE ASSIGN $s_i$ the value $s_{i-1} + l_S$
                END IF
            ELSE

```
                IF t_{i+1} is equal to 1 THEN ASSIGN s_i the value s_{i-1} + l_T
                ELSE ASSIGN s_i the value s_{i-1} + l_S
                END IF
            END IF
        ELSE
            IF (i is equal to 2) or ((i is greater than 2) and (t_{i-2} is not equal to 2)) THEN
                                ASSIGN s_i the value s_{i-1} + l + l_H
            ELSE ASSIGN s_i the value s_{i-1} + l_S + l_H
            END IF
        END IF
    END IF
END FOR
IF t_n is equal to 0 THEN ASSIGN s_n the value L
ELSE ASSIGN s_n the value L - l_E
END IF
END FUNCTION
```

```
FUNCTION calculating_hinge_coordinates()
INPUT: the topological code of beam: n-element sequence t; n coordinates of supports: n-element
                        vector s; the length l_H
OUTPUT: n - 2 coordinates of hinges: (n - 2)-element vector h
FOR i starts at 1, i < n - 1, increment i DO
    IF t_{n+1} is equal to 0 THEN ASSIGN h_n the value s_{n+1}
    ELSE IF t_{n+1} is equal to 2 THEN ASSIGN h_n the value s_{n+1} - l_H
    ELSE ASSIGN h_n the value s_{n+1} + l_H
    END IF
END FOR
END FUNCTION
```

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