JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
52, 1, pp. 81-92, Warsaw 2014

INFLUENCE OF SUPPORT COMPLIANCES ON STABILITY AND LIMITING SLENDERNESS RATIO OF COMPRESSION HELICAL SPRINGS

Jacek Krużelecki, Damian Szubartowski
Cracow University of Technology, Institute of Applied Mechanics, Kraków, Poland
e-mail: jacek.kruzelecki@pk.edu.pl; tszubartowski@o2.pl

In the paper, the influence of support compliances on the stability and the limiting slenderness ratio of compression helical springs was investigated. To analyze the stability of helical springs, the concept of an equivalent column was applied and a general system of supports described by three independent compliances was utilized. In the general case, the solution obtained, which describes the critical force, depends on two rotational compliances and a lateral compliance of a transverse shift of the support. The influence of the individual compliances on the critical load and on the limiting slenderness ratio was studied. Simple approximation formulas were proposed to describe these relationships.

Key words: helical spring, stability, support compliance, slenderness ratio

1. Introductory remarks

Helical springs, specially compression ones, fulfil different functions in the technique: do the work, store energy, alleviate the impact, raise or take vibration and many others. Moreover, they may be subjected to loss of stability.

A helical spring should be treated as a spatially curved bar. Such an approach, based on Kirchhoff-Clebsh equations and geometrically non-linear theory, was developed by Nikolai (1955). It was applied to the problems of stability of helical springs, among others, by Chernyshev (1946), Haringx (1948), Olenov (1977), Polishchuk (1977), Kernchen (1979). The equations of this type were also used by Ponomariev (1948), Taborrak and Xiong (1989) and others to derive equations of stability for the general behaviour of loadings. They are also used for determination of displacements in springs with high deformability. The current configuration of a spring under loading determined in this way can be treated as a precritical state (Chernyshev, 1958; Sutyrin, 1980). There are also attempts to build a finite element and, in this way, to solve the problem of stability taking into account geometrical nonlinearities. Let us mention here, for example, the paper by Mottershead (1982) on the dynamic stability. However, in most commercial finite element programs such an element does not exists. All the above papers deal with helical coil springs with a constant pitch.

Geometrically linear theory was developed by Trostel (1957) in connection with springs of arbitrary shape. This approach was used in the papers by Czerwiński (1973) – parabolic spring, Czerwiński (1975) – arbitrary geometry of a spring, Szafran-Gądek (1981) – elliptical spring, to determine displacements. Lin and Pisano (1988) studied the geometry of helical springs of arbitrary shapes.

An exact stability analysis of springs based on the theory of spatially curved bars is complicated and difficult in practical applications. Hence, in most engineering applications, a concept of an equivalent column is introduced. Such a column must account for compressibility of the axis and shear effect.

First papers using the concept of an equivalent column to stability analysis of helical springs are due to Hurlbrink (1910) and Grammel (1924). They took into account only the effect of change in the column length. They showed that there exists a limiting slenderness ratio below which no bifurcation is possible at all. Biezeno and Koch (1925) pointed out that the effect of shear should also be taken into account in the stability analysis of helical springs. However, they overestimated the impact of shear effects, and in consequence, concluded that any spring, even short, may be subjected to buckling. The correct solution, based on the concept of the equivalent column and assuming zero pitch angle (helix slope), was obtained by Haringx (1942), Ponomariev (1958) and others. These solutions have confirmed the existence of the limiting slenderness ratio, which was also verified experimentally. Suitable solutions can be found in several monographs, for example by Timoshenko and Gere (1961), Wahl (1963) and in application studies such as by Żukowski (1954) or Branowski (1989.

Differences between Biezeno's and Haringx-Ponomariev's solutions result from different estimations of the shear effect. This problem was investigated in detail by Ziegler (1982). He showed that for helical springs the so-called modified approach is the proper way of taking the shear effect into account, whereas Biezeno and Koch (1925) applied Engesser's approach.

Equivalent columns were also introduced for buckling analysis under combined compression and torsion. We can mention here papers by Ziegler and Huber (1950) and Satoh, Kunoh and Mizuno (1988).

In all the above papers, the study of stability was based on linear theory. Nonlinear theory of stability and equivalent column was used by Mizuno (1960) to study postcritical behavior of cylindrical springs with a constant pitch.

All the above mention papers are based on the simplifying assumption of a very small pitch angle: this angle was assumed to be zero. A more general equivalent column allowing for arbitrary helix slope is due to Berdychevsky and Sutyrin (1983), but that concept proposed for general nonlinear problems is rather complicated and has not been applied to stability analysis. Krużelecki and Życzkowski (1990) proposed a new, more refined and exact concept of an equivalent column for buckling of helical springs of arbitrary shape. It accounts for a helix angle and possible buckling in two planes and also takes into consideration axial compressibility and shear effects. It ensures high accuracy of results also. Such an equivalent column, described by compression, bending and shearing rigidities, gives the possibility to analyze stability of helical springs of arbitrary shape.

Most of the papers deal with simply supported or clamped springs at both ends. In this paper the influence of modes of supports and their compliances on stability and slenderness limiting ratio is examined. The concept of an equivalent column is used.

2. Equations of stability of compression helical springs

The concept of an equivalent column proposed by Hurlbrink (1910), Grammel (1924), Haringx (1942) and developed by Krużelecki and Życzkowski (1990) allows a helical spring to be treated in stability problems as a column with the appropriate rigidities. Because of much higher sensitivity of shearing on stability of a spring in comparison with a column having a solid cross-section, this effect should be also taken into account.

A helical spring – an equivalent column – with an initial length H_0 which is loaded by a compressive force P is considered. Because of high compressibility, a helical spring becomes shorter and its actual length is H. In that state described by large displacements with changes of geometry taken into account, a small bending described by a linear theory is imposed, and this way the critical state of the helical spring is investigated.

The differential equations of bucking of a spring for any type of support can be written in the following form

$$\frac{d\overline{w}}{dx} = \theta + \chi \qquad \frac{d\theta}{dx} = -\frac{M}{EI} \qquad \frac{dM}{dx} = Q + P\frac{d\overline{w}}{dx} \qquad \frac{dQ}{dx} = 0 \tag{2.1}$$

where x denotes the actual coordinate along the axis of the compressed spring $(0 \leqslant x \leqslant H)$, $\overline{w}$ is the lateral displacement, θ is the inclination of a normal to the cross-section against the central axis of the undeformed column (the angle of rotation of a cross-section caused by bending), M is the bending moment, Q represents the shear force in the spatial frame. In equation (2.1), the shearing effect is represented by the term χ according to the so-called modified approach discussed by Ziegler (1982)

$$\chi = \frac{Q + P\theta}{GA} \tag{2.2}$$

The actual bending EI, shearing GA and axial EA rigidities were defined in the paper by Krużelecki and Życzkowski (1990). It turned out that for the problem under consideration we can use a simplified model of an equivalent column, namely assume that the helix angle is equal to zero. Then, the current rigidities can be written as follows

$$EA = (EA)_0 = \frac{EI_wH_0}{2\pi(1+\nu)R_0^3n_0} \qquad EI = (EI)_0\frac{H}{H_0} \qquad GA = (GA)_0\frac{H}{H_0} \tag{2.3}$$

where

$$(EI)_0 = \frac{H_0EI_w}{\pi(2+\nu)R_0n_0} \qquad (GA)_0 = \frac{H_0EI_w}{\pi R_0^3n_0} \tag{2.4}$$

and $I_w = \pi d^4/64$, d is the diameter of the wire, R_0 and n_0 denote the initial radius and the original number of coils of the spring, respectively, E stands for the Young modulus and ν is the Poisson ratio. Such formulas describing the rigidities of a spring with the helix angle equal to zero can be also found in the papers by Haringx (1948), Żukowski (1954), Timoshenko and Gere (1961). The set of differential equations (2.1) of the first order can be transformed into one equation of the forth order

$$\frac{d^4\overline{w}}{dx^4} + k^2\frac{d^2\overline{w}}{dx^2} = 0 \tag{2.5}$$

for which the general solution can be written in the form

$$\overline{w}(x) = A\sin kx + B\cos kx + Cx + D \tag{2.6}$$

where

$$k = \sqrt{\frac{P}{EI}\Big(1 + \frac{P}{GA}\Big)} \tag{2.7}$$

Helical compression springs used in different technical applications are supported and fastened in many different ways. Influence of coil spring mountings on their selected operational properties are presented in the book by Michalczyk and Slawiński (2011). Analysis of modes of supports leads to the conclusion that the ends of a spring should be supported elastically taking one end as a support immovable in the space. Such a general way of support, presented in Fig. 1, can describe possible behavior of the ends of a real spring. A similar attempt to the system of supports for analysis of stability of solid columns was applied by Życzkowski (1988). The coordinate system is associated here with the bottom support which is treated as immovable one. The quantities C_1 and C_2 in Fig. 1 characterize the rotational compliances of the lower and upper supports, respectively while C_3 describes the lateral compliance of a transverse shift

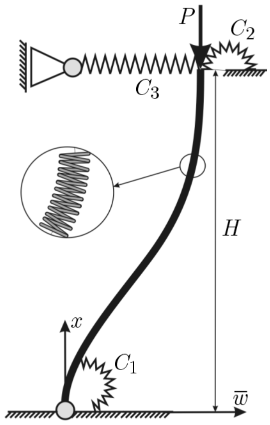

Fig. 1. General case of the spring support

of the upper end. So, assuming, for example $C_1 = 0$, it denotes that rotation of the lower end of the spring is completely blocked whereas $C_1 = \infty$ means a simply supported end. On the other hand $C_3 = 0$ denotes that lateral movement of the upper end is completely blocked while $C_3 = \infty$ indicates that later displacement of this end is free.

The boundary conditions for the considered general system of supports in the actual configuration (compressed spring) can be written as follows

$$\overline{w}(0) = 0 \qquad -C_1M(0) = \theta(0) \qquad C_2M(H) = \theta(H) \qquad C_3Q(H) = -\overline{w}(H) \tag{2.8}$$

Taking into account equations (2.1) and (2.2), boundary conditions (2.8) take the form

$$\begin{aligned}
&\overline{w}(0) = 0 \qquad C_1EI\overline{w}''(0) - \overline{w}'(0) - \overline{w}(H)\frac{1}{C_3GA} = 0\\
&C_2EI\overline{w}''(H) + \overline{w}'(H) + \overline{w}(H)\frac{1}{C_3GA} = 0\\
&C_3EI\overline{w}'''(H) + C_3P\Big(1 + \frac{P}{GA}\Big)\overline{w}'(H) - \Big(1 + \frac{P}{GA}\Big)\overline{w}(H) = 0
\end{aligned} \tag{2.9}$$

Boundary conditions (2.9) lead to four homogeneous algebraic equations for which the condition of non-trivial solution gives

$$\begin{aligned}
&\sqrt{\frac{H^2P}{EI}\Big(1 + \frac{P}{GA}\Big)}\Big\{\Big(\frac{C_3P}{H} - 1\Big)\Big[\Big(1 + \frac{P}{GA}\Big)^{-1} - EIC_1C_2P\Big]\\
&\quad + (C_1 + C_2)\frac{EI}{H}\Big\}\sin\left(\sqrt{\frac{H^2P}{EI}\Big(1 + \frac{P}{GA}\Big)}\right)\\
&\quad - \Big[2 - PH(C_1 + C_2)\Big(\frac{C_3P}{H} - 1\Big)\Big]\cos\left(\sqrt{\frac{H^2P}{EI}\Big(1 + \frac{P}{GA}\Big)}\right) + 2 = 0
\end{aligned} \tag{2.10}$$

Equation (2.10) is written in the carrent configuration, which means that it is expressed by the actual rigidities (GA) and (EI) and the actual (after copression) length H of the spring. Since, the critical loading P is "hidden", in the actual rigidities as well as in the actual length of the spring one has to express (2.10) by the initial quantities. Taking into accout that

$$H = H_0\Big(1 - \frac{P}{(EA)_0}\Big) \tag{2.11}$$

and utilizing (2.3) and (2.4), we obtain

$$
\begin{aligned}
&\lambda\sqrt{p_{cr}\frac{2+\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)}\Big\{[1-(1+\psi_3)p_{cr}]\Big[\lambda^2 p_{cr}\psi_1\psi_2\frac{2+\nu}{2+2\nu}-\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)^{-1}\Big]\\
&\quad+\psi_1+\psi_2\Big\}\sin\left(\lambda\sqrt{p_{cr}\frac{2+\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)}\right)\\
&\quad-\Big\{2+\lambda^2 p_{cr}(\psi_1+\psi_2)[1-(1+\psi_3)p_{cr}]\frac{2+\nu}{2+2\nu}\Big\}\\
&\quad\cdot\cos\left(\lambda\sqrt{p_{cr}\frac{2+\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)}\right)+2=0
\end{aligned}
\tag{2.12}
$$

where

$$
\begin{aligned}
&\lambda=\frac{H_0}{R_0} &\qquad& p_{cr}=\frac{P_{cr}}{(EA)_0} &\qquad& \psi_1=C_1\frac{(EI)_0}{H_0}\\
&\psi_2=C_2\frac{(EI)_0}{H_0} && \psi_3=C_3\frac{(EI)_0}{H_0^3} &&
\end{aligned}
\tag{2.13}
$$

and λ denotes the slenderness ratio of the spring, p_{cr} is the dimensionless critical force and ψ_i means the dimensionless compliance of supports. Equation (2.12) allows one to determine the critical force in terms of dimensionless compliances of supports. In (2.12), the rotational compliances ψ_1 and ψ_2 occur in pairs. It means that this equation is symmetrical with respect to these quantities and it also means that the replacement of ψ_1 by ψ_2 and vice versa does not cause change in the critical loading. The critical loading depends also on the slenderness ratio λ; for larger values of λ the critical force is smaller. If the slenderness ratio is small enough, the buckling of the spring does not occur. It means that the relation $p_{cr}=p_{cr}(\lambda)$ has its own return point which corresponds to the limiting slenderness ratio λ_{lim} below which no buckling occurs. Equation (2.12) also allows one to determine such limiting slenderness ratios for any mode of supports (for any values of compliances).

3. Support compliances vs. stability of helical springs

We analyze here relationships between modes of the support expressed by support compliances and stability of compressed helical springs. We limit our considerations to some chosen variants of the supports for which the classical fittings of ends of the springs constitute particular cases of the supports. Analysis is based on general solution (2.12) obtained via application of the equivalent column concept.

3.1. Rotational compliances

In this case, we analyze the influence of the rotational compliances ψ_1 and ψ_2 on the stability and on the limiting slenderness ratio of a helical compression spring. Since equation (2.12) is symmetrical with respect to ψ_1 and ψ_2, the dependence of the limiting slenderness ratio on these compliances should be common for ψ_1 and ψ_2. To find appropriate relationships, we assumed the values of ψ_3, namely $\psi_3=\infty$ or $\psi_3=0$ which refer to free movement and to completely blocked movement of the upper end of the spring, respectively, whereas ψ_1 and ψ_2 varies between the limiting values, $\infty\leqslant\psi_i\leqslant 0$, where $i=1$ or 2. That leads to analysis of three systems of supports for which three pairs of the particular (limiting) cases of supports are presented in Fig. 2.

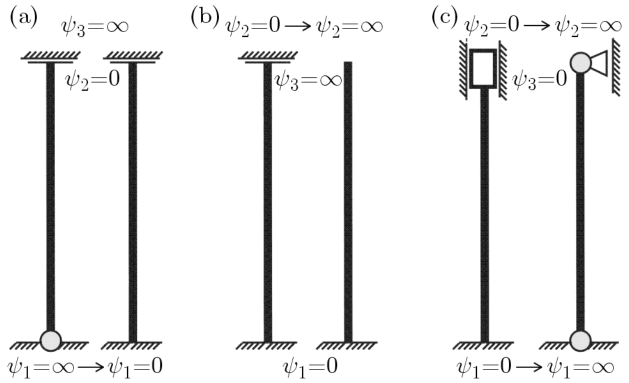

Fig. 2. Particular cases of the supports for analysis of the influence of ψ_1 or ψ_2

In the first case, it is assumed that only ψ_1 can vary ($\infty \leqslant \psi_1 \leqslant 0$) whereas $\psi_2 = 0$ and $\psi_3 = \infty$. The particular systems of supports for the limiting values of compliances referring to this case are presented in Fig. 2a. In the second case, we examine the influence of ψ_2 on p_{cr} and λ_{lim}. It means that only ψ_2 can vary ($0 \leqslant \psi_2 \leqslant \infty$) whereas $\psi_1 = 0$ and $\psi_3 = \infty$. The limiting systems of supports are shown in Fig. 2b. In the third case, presented in Fig. 2c, we analyze the influence of ψ_1 and ψ_2 on the assumption that $\psi_3 = 0$. As in the first and second case, the same $\psi_3 = \infty$ is assumed, and we can consider these two cases together. Substituting $\psi_2 = 0$ and $\psi_3 = \infty$ or substituting $\psi_1 = 0$ and $\psi_3 = \infty$ into (2.12) and assuming that only the first mode of buckling is considered, equation (2.12) can be rewritten as follows

$$\lambda\sqrt{p_{cr}\frac{2+\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)} = \pi - \arctan\left(\psi_i\lambda\sqrt{p_{cr}\frac{1+2\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)}\right) \tag{3.1}$$

where $i = 1$ or 2. In the limit case $\psi_i = \infty$, we obtain

$$p_{cr} = \frac{1+\nu}{1+2\nu}\left(1 \pm \sqrt{1-\frac{1+2\nu}{2+\nu}\frac{\pi^2}{\lambda^2}}\right) \qquad \lambda_{lim} = \pi\sqrt{\frac{1+2\nu}{2+\nu}} \tag{3.2}$$

whereas for $\psi_i = 0$ we have

$$p_{cr} = \frac{1+\nu}{1+2\nu}\left(1 \pm \sqrt{1-4\frac{1+2\nu}{2+\nu}\frac{\pi^2}{\lambda^2}}\right) \qquad \lambda_{lim} = 2\pi\sqrt{\frac{1+2\nu}{2+\nu}} \tag{3.3}$$

In Fig. 3a, curves $p_{cr} = p_{cr}(\lambda)$ for the considered particular cases of supports are presented where the upper parts of the curves, marked by dashed lines, refer to the regions above the return points (marked with circles in Fig. 3a). The upper parts of the curves do not describe the real buckling loadings. For $0 \leqslant \psi_i \leqslant \infty$, the curves can be obtained numerically from (3.1) and they are located between the limiting cases. Such a curve obtained for $\psi_i = 0.8$ is also presented in Fig. 3a. All these curves have the return points for the same value of p_{cr}, namely for $p_{cr} = (1+\nu)/(1+2\nu)$ but for different values of λ_{lim} which depend on values of ψ_i. The relationship $\lambda_{lim} = \lambda_{lim}(\psi_i)$ can be obtained numerically from equation (3.1) and it is presented in Fig. 3b (full line). That curve can be approximated with a very good accuracy by

$$\lambda_{lim} = \frac{0.9+0.56\nu-0.16\nu^2}{0.4+\psi_i} + \pi\sqrt{\frac{1+2\nu}{2+\nu}} \qquad i = 1,2 \tag{3.4}$$

and is shown in Fig. 3b by the dashed line (both lines overlap).

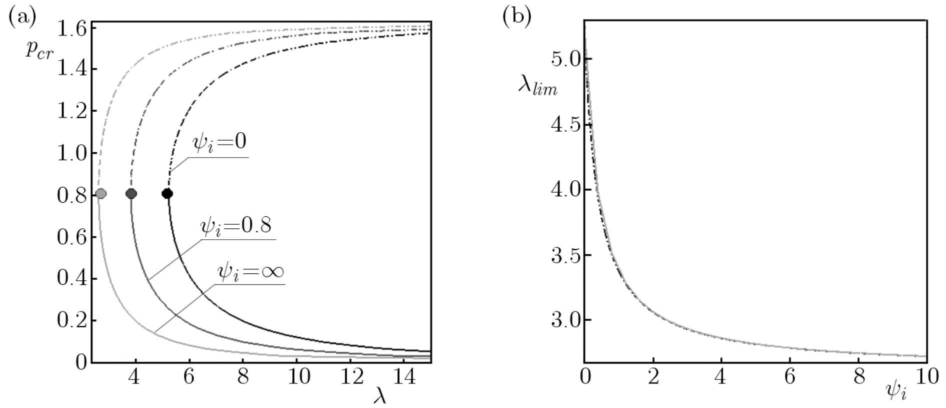

Fig. 3. Critical force vs. slenderness ratio (a), limiting slenderness ratio vs. rotational compliance ψ_i for $\nu = 0.3$ and $\psi_3 = \infty$

The third case, presented in Fig. 2c, shows a study of the impact ψ_1 and ψ_2 on the stability and the limiting slenderness ratio of the spring on the assumption that $\psi_3 = 0$. Substituting $\psi_3 = 0$ into (2.12) and assuming that only the first mode of buckling is considered, equation (2.12) can be rewritten as follows

$$\begin{aligned}
&\lambda\sqrt{p_{cr}\frac{2+\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)}\Big\{(1-p_{cr})\Big[\lambda^2 p_{cr}\psi_1\psi_2\frac{2+\nu}{2+2\nu}-\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)^{-1}\Big]\\
&\quad+\psi_1+\psi_2\Big\}\sin\Bigg(\lambda\sqrt{p_{cr}\frac{2+\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)}\Bigg)\\
&\quad-\Big[2+\lambda^2 p_{cr}(\psi_1+\psi_2)(1-p_{cr})\frac{2+\nu}{2+2\nu}\Big]\cos\Bigg(\lambda\sqrt{p_{cr}\frac{2+\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)}\Bigg)+2=0
\end{aligned} \tag{3.5}$$

For the limiting values of ψ_i ($\psi_i = 0$ and $\psi_i = \infty$), closed solutions (3.4) can be written as follows:
— for $\psi_1 = \psi_2 = 0$

$$p_{cr} = \frac{1+\nu}{1+2\nu}\Bigg(1 \pm \sqrt{1-16\frac{1+2\nu}{2+\nu}\frac{\pi^2}{\lambda^2}}\Bigg) \qquad \lambda_{lim} = 4\pi\sqrt{\frac{1+2\nu}{2+\nu}} \tag{3.6}$$

— for $\psi_1 = \psi_2 = \infty$

$$p_{cr} = \frac{1+\nu}{1+2\nu}\Bigg(1 \pm \sqrt{1-4\frac{1+2\nu}{2+\nu}\frac{\pi^2}{\lambda^2}}\Bigg) \qquad \lambda_{lim} = 2\pi\sqrt{\frac{1+2\nu}{2+\nu}} \tag{3.7}$$

For $0 < \psi_i < \infty$, only numerical solutions to (3.5) can be obtained. Such a sample solution for $\psi_i = 0.8$ together with solutions (3.6) and (3.7) for the limiting cases is presented in Fig. 4a.

All curves in Fig. 4a have the return points for the same value of p_{cr}, namely for $p_{cr} = (1+\nu)/(1+2\nu)$, but for different values of λ_{lim} which depend on values of ψ_i. The return points are marked with circles in Fig. 4a. The relation $\lambda_{lim} = \lambda_{lim}(\psi_i)$ was obtained numerically from (3.5) and is presented in Fig. 4b (full line). That curve is approximated with a very good accuracy by

$$\lambda_{lim} = \frac{0.89+0.53\nu-0.11\nu^2}{0.2+\psi_i} + 2\pi\sqrt{\frac{1+2\nu}{2+\nu}} \qquad i = 1,2 \tag{3.8}$$

and is shown in Fig. 4b by dashed line (both lines overlap).

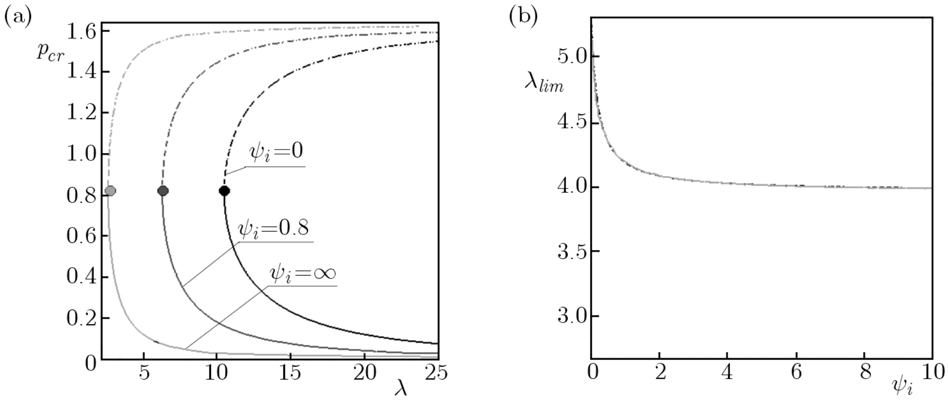

Fig. 4. Critical force vs. slenderness ratio (a), limiting slenderness ratio vs. rotational compliance ψ_i for $\nu = 0.3$ and $\psi_3 = 0$ (b)

3.2. Lateral compliance

In this case, we analyze the influence of the lateral compliance ψ_3 $(0 \leqslant \psi_3 \leqslant \infty)$ on the stability and the limiting slenderness ratio of the spring on the assumption that $\psi_1 = 0$ and $\psi_2 = \infty$ or $\psi_1 = \infty$ and $\psi_2 = 0$. It refers to analysis of two systems of supports for which two pairs of the limiting cases of the supports are presented in Fig. 5.

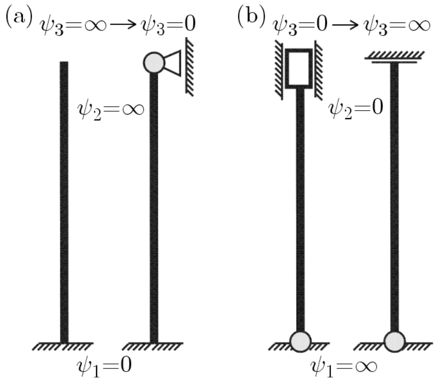

Fig. 5. Particular cases of the supports for analysis of the influence of ψ_3

Because general equation (2.12) is symmetrical with respect to ψ_1 and ψ_2, both cases can be considered together. Substituting into (2.12) $\psi_1 = 0$, $\psi_2 = \infty$ or $\psi_1 = \infty$, $\psi_2 = 0$ for the first mode of buckling, one obtains

$$\lambda\sqrt{p_{cr}\frac{2+\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)} = \pi+\arctan\left(\lambda\frac{1-(1+\psi_3)p_{cr}}{1-\frac{1+2\nu}{2+2\nu}p_{cr}}\sqrt{p_{cr}\frac{1+2\nu}{2+2\nu}\Big(1-\frac{1+2\nu}{2+2\nu}p_{cr}\Big)}\right) \tag{3.9}$$

The numerical solutions to (3.9) are presented in Fig. 6 for different values of ψ_3 $(0 \leqslant \psi_3 \leqslant \infty)$ and $\nu = 0.3$. These curves, in general, are different from those presented for the previous cases. Decreasing the compliance ψ_3 from infinity to a finite value, the systems of supports presented in Fig. 5a become more stiff. It causes that the critical force increases, especially for smaller values of ψ_3 and any value of λ. Simultaneously, the return points can move towards larger values of p_{cr} and λ. On the other hand, the shear force, which becomes larger for smaller values of ψ_3, has destabilizing influence on the stability of the spring and it causes that the critical force can decrease for any value of the slenderness ratio. It can result in changing of locations of the return points, namely they can move towards smaller values of p_{cr} and λ. These two different effects cause the curves have the inflexion points for the lateral compliance ψ_3 whose values are closer to $\psi_3 = 0$. Moreover, the return points, marked by circles in Fig. 6, move towards larger values of p_{cr} and λ for smaller values of ψ_3. So, in this case not only the limiting slenderness ratio but also the critical force of the return points depends on the lateral compliance ψ_3.

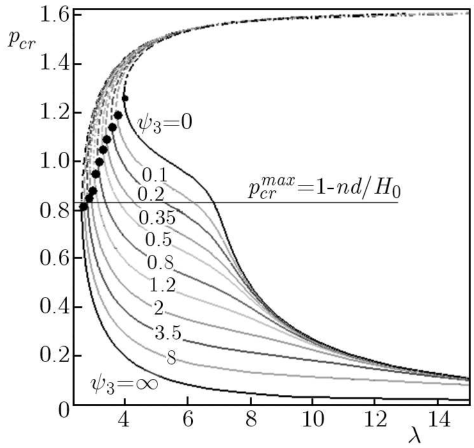

Fig. 6. Critical force vs. slenderness ratio for $\nu = 0.3$, $\psi_1 = 0$, $\psi_2 = \infty$ or $\psi_1 = \infty$, $\psi_2 = 0$

The largest dimensionless critical force, defined by $(2.13)_2$, and whose physical interpretation is the critical compressive strain, theoretically (diameter of the wire is equal to zero) can reach a value of 1, $p_{cr}^{max} = 1$. For larger values of p_{cr} than 1, the coils of the spring are in contact one with another (spring is closed). However, taking into account that diameter the wire is of a finite dimension, then the maximum p_{cr} is lower, namely $p_{cr}^{max} = 1 - nd/H_0$, where n denotes the number of coils and d is the diameter of the wire. This horizontal line, shown in Fig. 6, separates the admissible solutions (lines below) from those which are not possible to obtain (lines above). This line shows also that the return points can be reached only for large enough values of ψ_3, $\psi_3 \geqslant \overline{\psi}_3$.

The critical force for the return points vs. the lateral compliance ψ_3 can be approximated with a very good accuraccy by

$$p_{cr} = \frac{0.4}{0.9 + \psi_3} + \frac{1 + \nu}{1 + 2\nu} \qquad \text{for} \qquad \psi_3 \geqslant \overline{\psi}_3 \tag{3.10}$$

whereas the limiting slenderness ratio for the return points can be described by

$$\lambda_{lim} = \frac{0.47 + 0.49\nu + 0.13\nu^2}{0.61 - 0.66\nu + 0.63\nu^2 + \psi_3} + \pi\sqrt{\frac{1 + 2\nu}{2 + \nu}} \qquad \text{for} \qquad \psi_3 \geqslant \overline{\psi}_3 \tag{3.11}$$

where

$$\overline{\psi}_3 = \frac{0.4(1 + 2\nu)}{\nu - (1 + 2v)\frac{nd}{H_0}} - 0.9 \tag{3.12}$$

and the above approximations are valid for $\nu \geqslant 0.3$. For the lateral compliance $\psi_3 < \overline{\psi}_3$, neither the critical force nor the limiting slenderness ratio can be specified because the neighbouring coils settle one on another and the stability of the fully compressed spring cannot be described in this way. In Fig. 7a, the critical force for the return points vs. the lateral compliance ψ_3 is presented, whereas in Fig. 7b, the limiting slenderness ratio is shown for $\nu = 0.3$ on the assumption that $d = 0$.

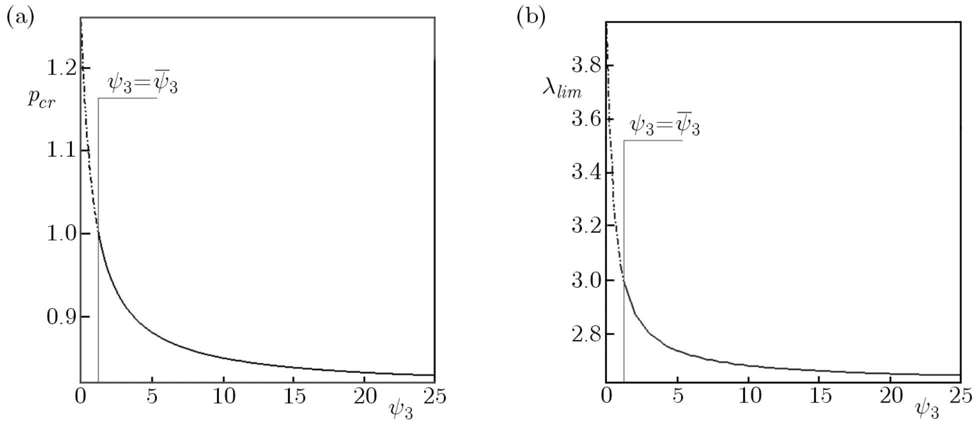

Fig. 7. Critical force (a), limiting slenderness ratio (b) vs. compliance ψ_3 for the return points, $\nu = 0.3$, $d = 0$

4. Final remarks

Analysis of the influence of support compliances on the stability and the limiting slenderness ratio showed that rotational compliances of the upper and lower ends of a compression helical spring have identical effects on the investigated quantities – algebraic equation (2.12) describing the critical load is symmetrical with respect to the rotational compliances ψ_1 and ψ_2. Each curve describing the critical force by both rotational compliances is located between the curves obtained for particular modes of the supports which were investigated in details. The values of the critical forces for the return points are the same for any value of the rotational compliances and it depends on the Poisson ratio. The values of the limiting slenderness ratio, below which the spring never buckles, also depend on the rotational compliances. Those relationships have a hyperbolic character and the appropriate approximation formulas are proposed in the paper.

Analysis of the influence of the lateral compliance showed that a decrease in the lateral compliance ψ_3 causes an increase in the shearing force. It has a significant impact on the stability and the limiting slenderness ratio. The shearing force causes that curves describing the relationship between the critical force and slenderness ratio has an inflexion point. The theoretical return point moves towards larger values of the critical forces and larger limiting slenderness ratios. It causes that for a small enough lateral compliance, the spring earlier becomes a fully compressed spring (gaps between the coils disappear), then it loses its stability.

References

1. Berdychevsky V.L., Sutyrin V.E., 1983, A problem of an equivalent rod in nonlinear theory of springs (in Russian), *Prikladnaia Mathematica i Mekhanica*, **47**, 2, 238-248
2. Biezeno C.B., Koch J.J., 1925, Knickung von Schraubenfedern, *Zeitschrift für Angewandte Mathematik und Mechanik*, **5**, 279-280
3. Branowski B., 1989, *Metalowe elementy sprężyste*, PWN, Warszawa
4. Chernyshev N.A., 1946, Stability of compression helical springs (in Russian), [In:] *Novye Metody Rascheta Pruzhin*, C.D.Ponomariev (Edit.), Moscow: Gosud. Nauk.-Tech. Lzd. Mashinnostr. Liter., 57-78
5. Chernyshev N.A., 1958, Nonlinear theory of deformation of helical springs (in Russian), [In:] *Problems of Strength of Materials*, Moscow: Mashgiz., **3**, 19-39
6. Czerwiński P., 1973, Stan odkształcenia śrubowej sprężyny paraboidalnej, *Archiwum Budowy Maszyn*, **22**, 2, 275-291

7. Czerwiński P., 1975, Przyczynek do wyznaczenia stanu odkształcenia sprężyn o dowolnej geometrii, *Rozprawy Inżynierskie*, **23**, 1,143-155
8. Grammel R., 1924, Die Knickung von Schraubenfedern, *Zeitschrift für Angewandte Mathematik und Mechanik*, **4**, 384-389
9. Haringx J.A., 1942, On the buckling and the lateral rigidity of helical springs, *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen*, **45**, 533-539, 650-654
10. Haringx J.A., 1948, On highly compressible helical springs and rubber rods, and their application for vibration-free mountings I, *R94, Philips Research Report*, **3**, 401-449
11. Haringx J.A., 1949, On highly compressible helical springs and rubber rods, and their application for vibration-free mountings II, *R101, Philips Research Report*, **4**, 49-90
12. Hurlbrink E., 1910, Berechnung zylindrischer Druckfedern auf Sicherheit gegen seitliches Ausknicken, *Zeitschrift für Angewandte Mathematik und Mechanik*, **54**, 133-137, 181-184
13. Kernchen E., 1979, Zur Stabilität von Schraubenfedern, *Ingenieur-Archiv*, **48**, 382-392
14. Krużelecki J., 1992, *Problematyka optymalnego kształtowania ściskanych sprężyn śrubowych przy warunkach stateczności*, Monografia 143, Wydawnictwo Politechniki Krakowskiej, Kraków
15. Krużelecki J., Życzkowski M., 1990, On the concept of an equivalent column in the problem of stability of compressed helical springs, *Ingenieur-Archiv*, **60**, 367-377
16. Lin Y., Pisano A.P., 1988, The differential geometry of the general helix as applied to mechanical springs, *Journal of Applied Mechanics*, **55**, 12, 831-836
17. Michalczyk K., J. Slawiński J., 2011, *Wpływ sposobu mocowania sprężyn śrubowych na ich wybrane własności eksploatacyjne*, Wydawnictwo AGH, Kraków
18. Mizuno M., 1960, Problem of large deflection of coiled springs, *Bulletin of JSME*, **3**, 9, 95-103
19. Mottershead J.E., 1982, The large displacements and dynamic stability of springs using helical finite elements, *International Journal of Mechanical Sciences*, **24**, 547-558
20. Nikolai E.L., 1955, On problems of spatially curved elastic bar (in Russian), [In:] *Trudy po mekhanike*, E.L. Nikolai (Edit.), Moscow: Gosud. Izd. Tech.-Teoret. Liter., 45-277
21. Olenov E.L., 1977, Approximative method of calculation of critical load for multiwire helical springs under compression (in Russian), *Izv. Vysh. Ucheb. Zav., Mashinostroenie*, **9**, 9-14
22. Polishchuk D.F., 1977, Elastic stability of helical springs, *Dinamika, Prochnost i Dolgovechnost Detaley Mashin*, **2**, 38-47
23. Ponomariev S.D., 1948, Stability of helical springs under compression and torsion (in Russian), [In:] *Mashinostroenie*, E.A. Chudakov (Edit.), Moscow: Mashgiz, 638-685
24. Ponomariev S.D. (Edit), 1958, *Współczesne metody obliczeń wytrzymałościowych w budowie maszyn*, Cz.2, PWN, Warszawa, 58-79, 83-201
25. Satoh T., Kunoh T., Mizuno M., 1988, Buckling of coiled springs by combined torsion and axial compression, *JSME International Journal. Ser. 1, Solid Mechanics, Strength of Materials*, **31**, 56-62
26. Sutyrin V.G., 1980, On equations of theory of springs (in Russian), *Doklady Akademii Nauk SSSR*, **155**, 1, 60-64
27. Szafran-Gądek H., 1981, Metoda obliczania sprężyn śrubowych walca eliptycznego, *Prace Komisji Mechaniki Stosowanej PAN-Kraków, Mechanika*, **11**, 63-81
28. Tabarrok B., Xiong Y., 1989, On the buckling equation for spatial rods, *International Journal of Mechanical Sciences*, **31**, 3, 179-192
29. Timoshenko S.P., Gere J.M., 1961, *Theory of Elastic Stability*, 2nd ed. New York, Toronto, London: McGraw Hill

30. Trostel R., 1957, Beitrag zur Berechnung räumlich gekrümmter Stäbe nach der Theorie erster Ordnung, *Ingenieur-Archiv*, **25**, 414-423

31. Wahl A., 1963, *Mechanical Springs*, New York, Toronto, London: McGraw Hill

32. Ziegler H., 1982, Arguments for and against Engesser's formulas, *Ingenieur-Archiv*, **52**, 105-113

33. Ziegler H., Huber A., 1950, Zur Knickung der gedruckten und tordierten Schraubenfeder, *Zeitschrift für angewandte Mathematik und Physik*, **1**, 183-195

34. Żukowski S., 1954, *Sprężyny*, PWT, Warszawa

35. Życzkowski M., 1988, *Stateczność prętów i ustrojów prętowych, Mechanika Techniczna, T.IX, Wytrzymałość elementów konstrukcyjnych*, PWN, Warszawa

Manuscript received March 28, 2013; accepted for print June 14, 2013