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FE MODEL FOR LINEAR-ELASTIC MIXED MODE LOADING: ESTIMATION OF SIFS AND CRACK PROPAGATION

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Finite element analysis combined with the concepts of linear elastic fracture mechanics provides a practical and convenient means to study the fracture and crack growth of materials. The onset criterion of crack propagation is based on the stress intensity factor, which is the most important parameter that must be accurately estimated and facilitated by the singular element. The displacement extrapolation technique is employed to obtain the SIFs at crack tip. In this paper, two different crack growth criteria and the respective crack paths prediction for several test cases are compared between the circumferential stress criterion and the strain energy density criterion. Several examples are presented to compare each criterion and to show the robustness of the numerical schemes.

Keywords: crack propagation, stress intensity factor, strain energy density

1. Introduction

Since the first studies in the early 1920s by Inglis (1913), Griffith (1920) and Irwin amd Washington (1957), the research in linear elastic fracture mechanics has led to the development of a vast number of theories and applications. The propagation of a crack in a part leads to an important displacement discontinuity. A more accurate way of modelling such a discontinuity in a finite element mesh is to modify the part topology (due to crack propagation) and to perform automatic re-meshing. However, several methods have been proposed in the literature to model this discontinuity without any re-meshing. Belytschko *et al.* (1994) suggested a meshless method (Element free Galerkin method) where the discretisation is achieved by a model which consists of nodes and a description of the surfaces of the model. More recently, two interesting finite element techniques have been presented to deal with such discontinuities without any re-meshing stage. The Strong Discontinuity Approach (SDA) (Oliver et al, 2006; Dias-da-Costa *et al.*, 2009) in which displacement jumps due to the presence of the crack are embedded locally in each cracked finite element without affecting the neighbouring elements. The amplitude of displacement jumps across the crack are defined using additional degrees of freedom related to the plastic multiplier, leading to a formulation similar to standard non-associative plasticity models. The Extended Finite Element Method (X-FEM) (Möes *et al.*, 1999; Combescure *et al.*, 2008; Tran and Geniaut, 2012) in which the displacement-based approximation is enriched near a crack by incorporating both discontinuous fields and the near tip asymptotic fields through a Partition of Unity method (Babuska and Melenk, 1997; De Borst *et al.*, 2006). However, these recent techniques still have to be improved in order to deal with complex configurations such as multiple cracks, large deformation crack propagation, etc. When re-meshing is possible, a real mesh discontinuity representing the crack seems to be more accurate. This technique is used, in particular, in the 2D and 3D fracture numerical code FRANC (Carter *et al.*, 2000; Hernández-Gómez *et al.*, 2004). However, whatever the technique used, the accuracy of the crack path directly depends on the accuracy of kinking criteria.

In the literature, the comparative study of the propagation of one or multiple cracks in homogeneous or multi-materials structures, using the various criteria, is studied little. This paper presents a finite element analysis for the modeling of the crack growth problems using the displacement extrapolation technique. This modeling is based on the circumferential stress criterion and the strain energy density criterion to present a comparison of the crack propagation paths obtained for various applications.

For the goal to model the cracks propagation in multi-materials structures, examples of propagation in parts containing inclusions are studied. This characteristic can be very interesting in civil engineering for propagations in concrete, or for propagations in composite or multi-layer parts.

2. Crack growth criteria

In order to simulate crack propagation under linear elastic condition, the crack path direction must be determined. There are several methods used to predict the direction of the crack trajectory such as the maximum normal stress theory (or the maximum circumferential stress theory, Erdogan and Sih (1963)) and the minimum strain energy density theory (Sih, 1974).

2.1. Maximum circumferential stress criterion (MCSC)

This criterion, introduced by Erdogan and Sih (1963) for elastic materials, states that the crack propagates in the direction for which the circumferential stress $\sigma_{\theta\theta}$ is maximum. It is a local approach since the direction of crack growth is directly determined by the local stress field along a small circle of radius r centered at the crack tip.

The kinking angle θ of the propagating crack can be determined after calculating the values of the stress intensity factors K_I and K_{II}

$$\tan\frac{\theta}{2} = \left[\frac{1}{4}\frac{K_I}{K_{II}} \pm \frac{1}{4}\sqrt{\left(\frac{K_I}{K_{II}}\right)^2 + 8}\right] \tag{2.1}$$

where: K_I and K_{II} are respectively the stress intensity factors corresponding to mode I and mode II loading.

2.2. Minimum strain energy density criterion (MSEDC)

Sih (1974) postulated the critical value of the local strain energy as a criterion of crack instability. The minimum of strain energy density around the crack tip determines the direction of crack propagation. The angle of crack propagation θ can be determined by solving the following equations

$$\frac{\partial S}{\partial \theta} = 0 \qquad \frac{\partial^2 S}{\partial \theta^2} \geqslant 0 \tag{2.2}$$

where

$$S = \frac{dW}{dV}r \qquad S = \frac{1}{\pi r}(a_{11}K_I^2 + 2a_{12}K_1K_{II} + a_{22}K_{II}^2) \tag{2.3}$$

with

$$
\begin{aligned}
a_{11} &= \frac{1+\nu}{8E}[(3-4\nu-\cos\theta)(1+\cos\theta)] \\
a_{22} &= \frac{1+\nu}{8E}[4(1-\nu)(1-\cos\theta)+(1+\cos\theta)(3\cos\theta-1)] \\
a_{12} &= \frac{1+\nu}{8E}[2\sin\theta(\cos\theta-(1-2\nu))]
\end{aligned}
$$

where E is the modulus of elasticity, ν is Poisson's ratio, dW/dV is the elastic energy per unit volume V and S represents the intensity of the strain energy density.

3. Numerical calculation of stress intensity factors

The fracture mechanics is based on the determination of stress intensity factors. It is therefore important to develop a numerical model capable of calculating these factors for different geometries of cracked structures under different boundary conditions. In this paper, the displacement extrapolation method (Phongthanapanich and Dechaumphai, 2004) is used to calculate the stress intensity factors K_I and K_{II} as follows

$$
\begin{aligned}
K_I &= \frac{E}{3(1+\nu)(1+k)}\sqrt{\frac{2\pi}{L}}\Big[4(v_b-v_d)-\frac{v_c-v_e}{2}\Big] \\
K_{II} &= \frac{E}{3(1+\nu)(1+k)}\sqrt{\frac{2\pi}{L}}\Big[4(u_b-u_d)-\frac{u_c-u_e}{2}\Big]
\end{aligned}
\tag{3.1}
$$

where: $k=3-4\nu$ for plane strain and $k=(3-\nu)/(1+\nu)$ for plane stress, L is the length of the element side connected to the crack tip, u_i and v_i ($i=b,c,d$ and e) are the nodal displacements at nodes b, c, d and e in the x and y directions, respectively (see Fig. 1).

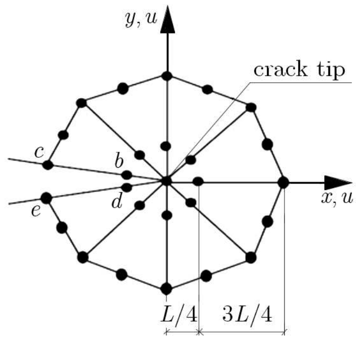

Fig. 1. Special elements used for the displacement extrapolation method

In order to obtain a better approximation of the field near the crack tip, special quarter point finite elements proposed by Barsoum (1977) are used, where the mid-side node of the element in the crack tip is moved to 1/4 of the length of the element L.

4. Numerical results and validation

4.1. Single edge cracked plate

This geometry is a rectangular plate (200 mm × 100 mm × 1 mm) with an initial crack (a = 30 mm) is considered for 2-dimensional finite element analysis (Fig. 2). The material properties of the aluminum alloy used in this study were taken as $E = 72\,400$ MPa, $\nu = 0.3$ and $K_{IC} = 1297\,\mathrm{N/mm^{3/2}}$, where E, ν and K_{IC} represents Young's modulus, Poisson's ratio and

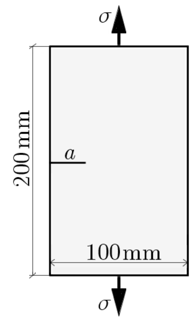

Fig. 2. The geometry of the single edge cracked plate

fracture toughness, respectively. The cracked plate is submitted under a uniform tensile load σ at both ends. The FE standard code ANSYS has been employed for modeling the problem.

For mesh generation of the cracked plate, the element type 'PLANE183' of ANSYS code is used, as shown in Fig. 3a. It is a higher order two-dimensional, 8-node element having two degrees of freedom at each node (translations in the nodal x and y directions), quadratic displacement behavior and the capability of forming a triangular-shaped element, which is required at the crack tip areas.

Due to the singular nature of the stress field in the vicinity of the crack, the singular elements shown in Fig. 5b, are considered at each crack tip area, which are modeled with a finer mesh.

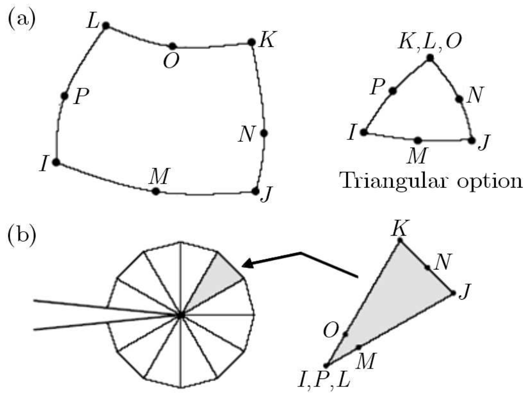

Fig. 3. (a) 'PLANE183' eight-node finite element and (b) singular option

A typical FE model of the cracked plate is shown in Fig. 4a. The special quarter point singular elements proposed by Barsoum are used for the modeling of the singular field near the crack tip (Fig. 4b). This mesh will serve to calculate the stress intensity factors K_I and K_{II} using the displacement extrapolation method implemented in ANSYS software.

The analytical stress intensity factor K_I for this problem is given by (Ewalds and Wanhill, 1989) as

$$K_I = F\sigma\sqrt{\pi a} \tag{4.1}$$

where F is the correction factor given by

$$F = 1.12 - 0.231\frac{a}{w} + 10.55\Big(\frac{a}{w}\Big)^2 - 21.72\Big(\frac{a}{w}\Big)^3 + 30.39\Big(\frac{a}{w}\Big)^4 \qquad \text{with} \qquad \frac{a}{w} \leqslant 0.6$$

The computed values of the stress intensity factor obtained by the displacement extrapolation method (Eq. $(3.1)_1$) under plane stress condition are compared with the numerical results using the J-integral method (Rice, 1968) and the analytical solutions (Eq. (4.1)) as shown in Fig. 5.

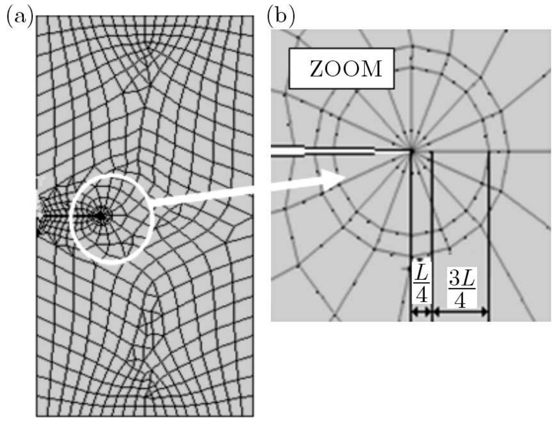

Fig. 4. (a) FE model of the cracked plate and (b) special elements used for the displacement extrapolation method

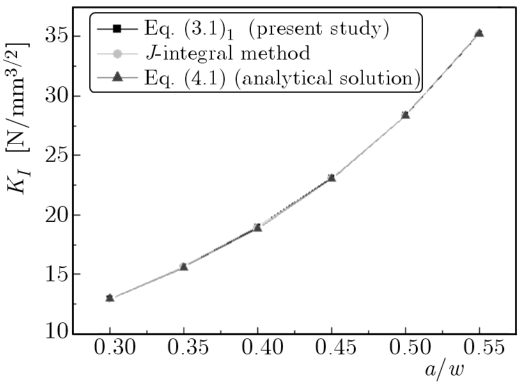

Fig. 5. Relationship between SIF K_I and the initial crack length size a/w

This comparison shows a good correlation between the three calculations. These results allow us to conclude that the numerical model used correctly describes the stress and deformation field near the crack tip, under pure mode I conditions.

Estimation of stress intensity factors by the extrapolation method leads us to determine the kinking angle of the MCS criterion using Eq. (2.1). This angle can also be determined by the MSED criterion by the mathematical resolution of Eqs. (2.2). Figure 6 illustrates the variation of the angle estimated at each increment the crack length. There is a very good correspondence between the results obtained by these approaches. In this example, the kinking angle θ varies between 0 and $-0.25°$, i.e. the crack propagates horizontally along the opening mode (mode I), as shown in Fig. 9.

Figure 7a,b shows the final step of crack propagation obtained with the MCS and MSED criterion. As expected, the crack propagates horizontally, depending on mode I loading. The propagation path is very regular and the concentric mesh keeps good accuracy at the crack tip. The results obtained allow us to conclude that the two criteria give a good crack propagation path under mode I loading.

Figure 7c,d,e shows the final configuration corresponding to the last evaluated crack length for the single edge cracked plate used by Alshoaibi and Ariffin (2006). The results obtained from the FRANC2D/L program and the results given by Alshoaibi and Ariffin (2006) using the adaptive mesh strategy are compared with those obtained from the FE software Ansys, using the MCS criterion. The behavior of crack propagation is almost the same as shown in this figure.

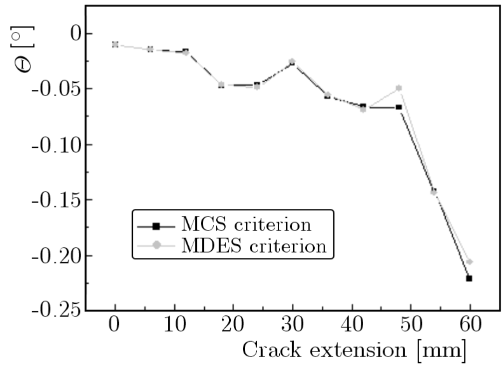

Fig. 6. Evolution of kinking angle θ during crack propagation, comparison between the MCS and MSED criterion

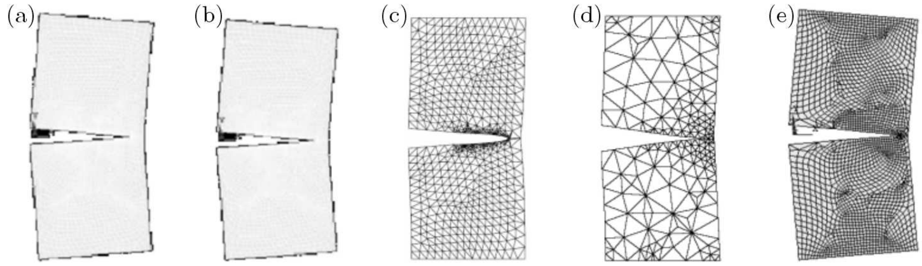

Fig. 7. Crack propagation path under mode I loading: (a) MCS criterion, (b) MSED criterion, (c) FRANC2D/L, (d) Alshoaibi and Ariffin (2006) results, (e) present study (MCS criterion)

4.2. Single edge angled crack

In what follows, we propose to study the propagation path of a crack inclined to the horizontal with the angle $\alpha = +20°$. The determination of stress intensity factors, angle of direction and crack growth path are made of plane stress problems and under the same loading conditions. A rectangular plate with an oblique crack and final mesh for the first step of the crack propagation are shown in Fig. 8.

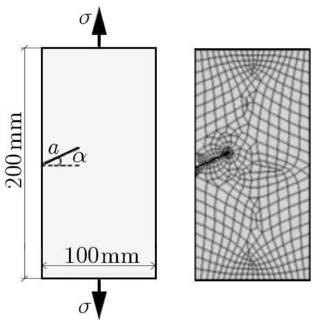

Fig. 8. Geometry and final mesh of a single edge angled crack

Figure 9 shows the evolution of stress intensity factors K_I and K_{II} during crack propagation steps obtained by the MCS criterion. These results are compared with those obtained by the MSED criterion. The plotted curves show a good agreement between these two approaches.

Figure 10a compares the crack trajectories of the MSED and MCSD criterion. The global trajectories are similar.

Figure 10b illustrates the variation of the kinking angle θ of the initial crack according to the inclination angle α. The results obtained show that the direction angle θ increases with the orientation angle α, and a slight difference $\Delta\theta$ occurs between the plotted curves by the

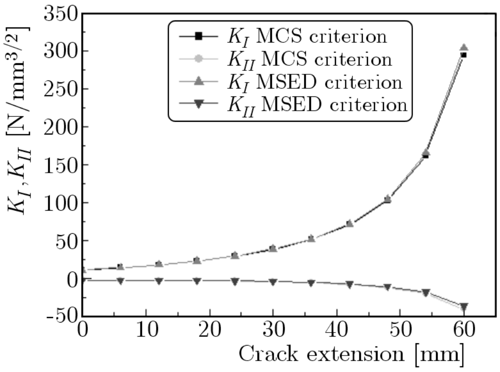

Fig. 9. Comparisons of SIFs for the single edge angled crack

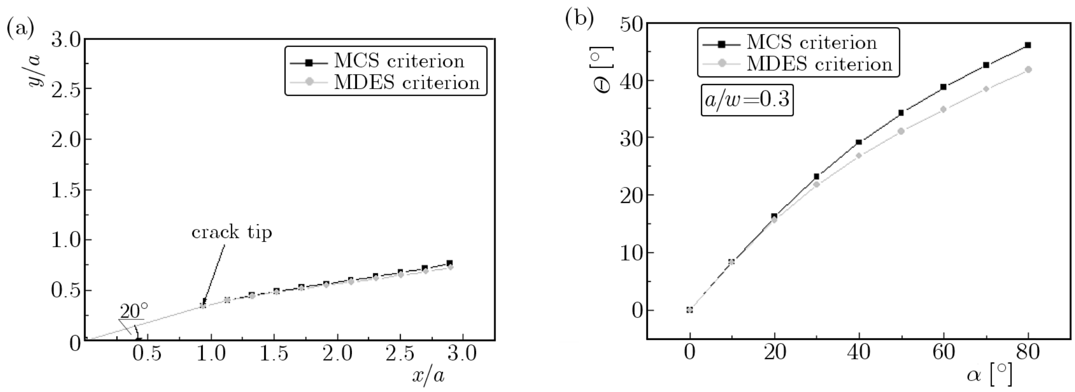

Fig. 10. (a) Crack trajectories comparison for the single edge angled crack. (b) Initial direction angle θ versus inclination angle α

two approaches. The maximum difference $\Delta\theta$ is about $4°$ when the angle α varies from $0°$ to $80°$. This difference in the first load step may involve important differences in the final crack trajectory for the examples whose geometry or loading is more complex. Bouchard (2000) and Bouchard *et al.* (2003) showed, for complex applications such as double edge cracked plate with two holes, that the MCS local criterion presents a more precise results than the MSED energy criterion, and an advantage of being able to be used for elastic-plastic materials (Bocca *et al.*, 1991; Baouch, 1998; Lebaillif and Recho, 2007). The MSED criterion, the accuracy is directly dependent on the number of elements in the ring around the crack tip (Bouchard *et al.*, 2003).

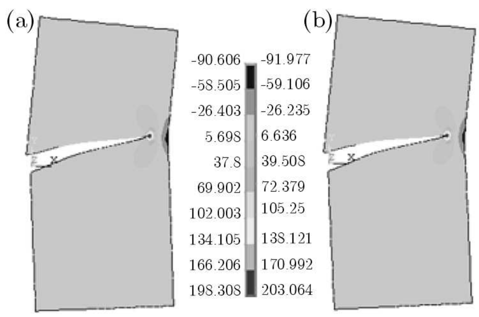

Fig. 11. Crack propagation path of the inclined crack with stress distribution contours predicted by: (a) MCS criterion and (b) MSED criterion

In Fig.11, one can visualize the final trajectory of the crack propagation estimated by both approaches and the contours of stress distribution σ_y [MPa]. The calculations show that each crack propagates with a deviation from the correct path towards the horizontal direction. This deviation is related to the choice of the crack increment length Δa. Over this distance the deviation is small and closer to reality.

4.3. Single edge cracked plate with one and two holes

In order to determine the effect of a geometrical defect on the crack propagation and to provide a comparison between the two theories, we chose two geometries of a single edge cracked plate with one and two holes. Figure 12 shows the dimensions [mm] and final mesh for the first step of crack propagation.

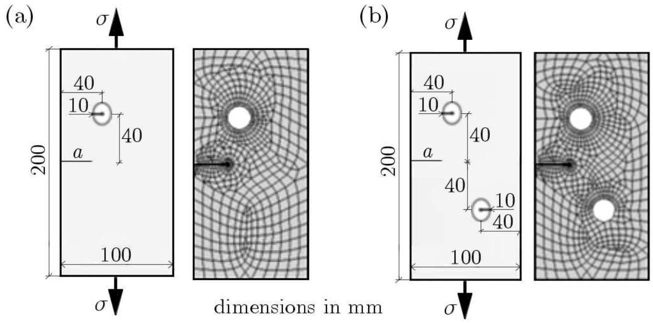

Fig. 12. Geometry and final mesh for the single edge cracked plate with: (a) off-center hole and (b) two holes

Figures 13a and 13b compare the variation of stress intensity factors K_I and K_{II} during crack propagation steps predicted for both plates. The plotted curves show a good agreement between the two theories. The values of SIFs K_I and K_{II} enable one to determine in Figs. 14a and 14b the final crack path by calculating the crack propagation direction at each time step. For the two geometries of the cracked plate, the results obtained show a good correlation between the trajectories estimated by these approaches.

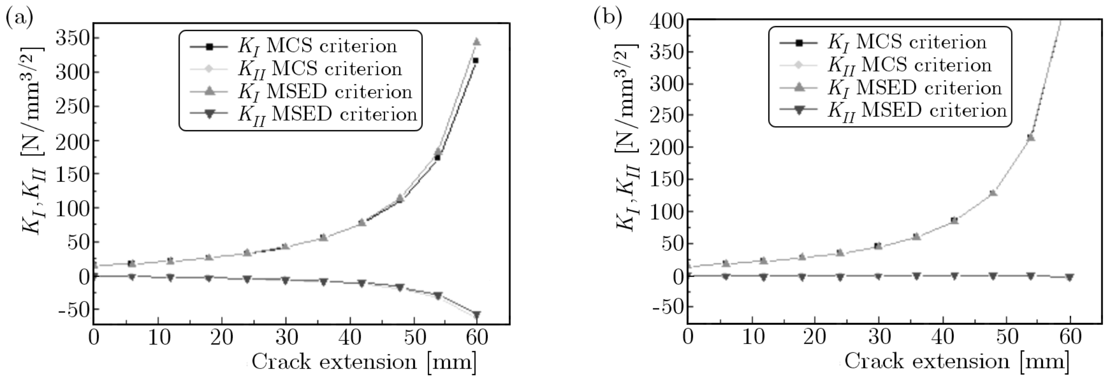

Fig. 13. Comparisons of SIFs for the single edge cracked plate with: (a) one hole and (b) two holes

Figures 13a and 13b show that in the case of crack propagation under mixed mode loading, the SIF K_I always takes positive values. Far from defect (Fig. 13a), SIF K_{II} is null when the inclination angle α is null, and decreases proportionally when the crack is directed towards the hole under the effect of stress concentration due to the discontinuity in the geometry, in the direction for which the circumferential stress $\sigma_{\theta\theta}$ is maximum (Erdogan and Sih, 1963). Using the MSED criterion (Sih, 1974), this orientation can be explained by the fact that the crack is directed towards the defect, in the direction for which the strain energy density (dW/dV) is minimal. This behavior of crack propagation is similar to the observed in experiments by Miranda *et al.* (2003), numerically by Boulenouar *et al.* (2012, 2013), Azocar *et al.* (2010), and analytically by Valentini *et al.* (1999).

Finally, we note that the sign of K_{II} should be opposite to the sign of the crack direction θ (Andersen, 1998). In our case, the SIF K_I can present positive values if the hole is positioned in the inferior part of the cracked plate.

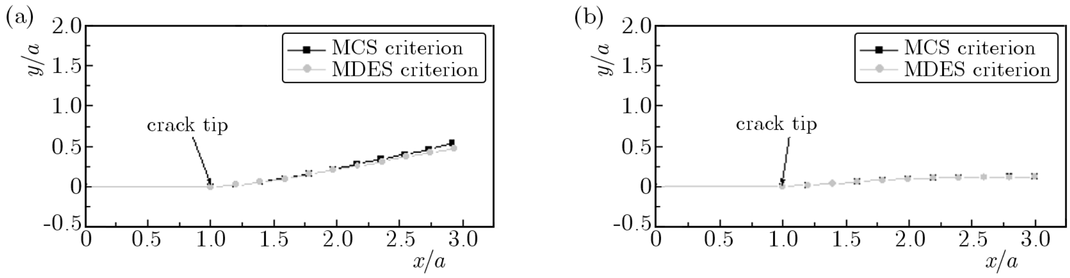

Fig. 14. Crack trajectories comparison for: (a) single edge cracked plate with one hole and (b) single edge cracked plate with two holes

In the case the plate contains only one hole, Fig. 15 illustrates the final step of crack propagation path obtained by the two theories. We note that each crack is moving towards the hole. This is because the hole creates a depression of stress that will change the maximum principal stress field in the plate and drew the crack. Once it exceeds the hole, this crack moves away slightly from the hole. These observations correspond well with those obtained by Bouchard *et al.* (2003) and Rashid (1998). Figure 15 also illustrates the distribution of the maximum stresses σ_y [MPa] in the plates, with a divergence between the results obtained by the two theories.

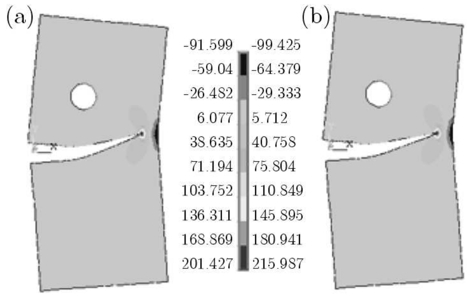

Fig. 15. Crack propagation path and stress distribution predicted by: (a) MCS criterion and (b) MSED criterion

Results of numerical simulations shown in Fig. 16 clearly the influence of the second hole on the crack propagation path predicted by both approaches. First of all, each crack directs slightly towards the first hole, then it reorientates horizontally under the influence of the second hole. These observations correspond well with those obtained by Lebaillif and Recho (2007).

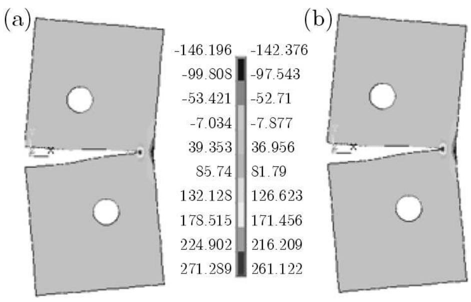

Fig. 16. Crack propagation path and stress distribution predicted by: (a) MCS criterion and (b) MSED criterion

5. Conclusion

In this paper, two different crack growth criteria and the crack paths prediction for various applications are compared using the maximum circumferential stress criterion and the strain energy density criterion. For each example, we determined the angle of direction at each crack increment length and the final path of crack propagation.

In order to obtain a better approximation of the field near the crack tip, special quarter point finite elements proposed by Barsoum (1977) are used.

The displacement extrapolation method is used to determine the stress intensity factors under mode I and mixed mode loading. Numerical calculations made by the finite element method show that this technique can correctly describe the stress and deformations field near the crack tip.

The two criteria give good results on the crack propagation path and the results between them are very close.

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