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BUCKLING OF HEATED TEMPERATURE DEPENDENT FGM CYLINDRICAL SHELL SURROUNDED BY ELASTIC MEDIUM

M. Sabzikar Boroujerdy
Department of Engineering, Firoozkooh Branch, Islamic Azad University, Firoozkooh, Iran
e-mail: mostafa.sabzikar@gmail.com

R. Naj
Research Institute of Petroleum Industry (RIPI), West Blvd. Azadi Sport Complex, Tehran, Iran
e-mail: najr@ripi.ir

Y. Kiani
Mechanical Engineering Department, Amirkabir University of Technology, Hafez Avenue, Tehran, Iran

Based on the Donnell theory of shells combined with the von-Karman type of geometrical nonlinearity, three coupled equilibrium equations for a through-the-thickness functionally graded cylindrical shell embedded in a two parameter Pasternak elastic foundation are obtained. Equivalent properties of the shell are obtained based on the Voigt rule of mixture in terms of a power law volume fraction for the constituents. Properties of the constituents are considered to be temperature dependent. The temperature profile through the shell thickness is obtained by means of the central finite difference method. Linear prebuckling analysis is performed to obtain the prebuckling forces of the cylindrical shell. Stability equations are derived based on the well-known adjacent equilibrium criterion. Three coupled partial differential stability equations are solved with the aid of a hybrid Fourier-GDQ method. After validating the numerical results, some parametric studies are conducted to investigate the influence of various parameters, especially foundation interaction. It is shown that elastic foundation enhances the critical buckling temperature difference of the shell and violates the buckled pattern.

Keywords: cylindrical shell, thermal buckling, Pasternak foundation, functionally graded materials

1. Introduction

As a class on novel structures, functionally graded materials have gain incredible attentions among researchers, specially when temperature loadings are also included in investigations. Cylindrical shells are a class of structures which are widely used in mechanical, civil and structural engineering. For cylindrical shells made of Functionally Graded Materials (FGMs), stability analysis under thermal loading is a major step for design.

When the edges are considered to be simply-supported, linear stability analysis of FGM cylindrical shells under thermal or mechanical loading is well-reported in the literature. In this case, closed-form solutions are available. This situation is accepted for simply-supported shells since both linear stability equations and boundary conditions accept the conventional Navier solution and pre-buckling deformations are confined to be membrane-like. For a large class of researches on linear thermal buckling of cylindrical shells, one may refer to the works reported by Shahsiah and Eslami (2003), Wu *et al.* (2005), Mirzavand and Eslami (2007) and Sofiyev (2007). However, these type of solutions are confined only to simply-supported edge conditions.

When in the pre-buckling deformation of the shell, bending deformations are also considered, in general critical buckling load/temperatures are not available in closed-form expressions.

In such a case, the non-linear equilibrium equations are solved and the critical buckling load/temperature is obtained through the load-deflection path of the structure. Researches of Shen (2004, 2007) and Shariyat (2008a,b) are categorized in this branch. It is well-accepted that linear buckling analysis of cylindrical shells results in an over-estimation in the critical-buckling load/temperature prediction, especially when the shell is short in length. However, since the solution methodology of this class of problems is more easily for a designer to follow and the obtained results are acceptable with an slight modification for design purposes, linear buckling analysis of shells is of high importance.

Researches on linear thermal stability of FGM cylindrical shells with clamped edges when compared with those of simply-supported conditions, are limited in number. The major reason is that, due to the clamped boundary conditions, closed-form solutions of linear stability equations are not available. In this category, when the nonlinear equilibrium equations are solved, results of Shen (2004, 2007) for clamped shells, and when the linear stability equations are solved, results of Sun *et al.* (2013) for arbitrary type of edge supports and of Kadoli and Ganesan (2006) for the clamped type of edge support are available in literature.

Researches on thermal stability analysis of cylindrical shells embedded in an elastic medium are limited in number. Investigations of Bagherizadeh *et al.* (2012) for linear thermal buckling of cylindrical shells in the Pasternak medium, Sofiyev (2011) for linear thermal buckling of conical shell in the Pasternak medium, Sheng and Wang (2008) for linear thermal buckling of cylindrical shells in a Winkler foundation and Shen (2013) for nonlinear thermal buckling and post-buckling of cylindrical shells are some of the available researches. The only examination considering the shell-foundation interaction, temperature dependency and heat conduction is the study by Shen (2013). The present research deals with thermal buckling of cylindrical shells in a two parameter elastic foundation with regard to temperature dependency of the constituents. The complete set of equations are obtained with the aid of the virtual displacements principle. The prebuckling solution is obtained with the aid of the linear membrane prebuckling solution. Stability equations are obtained by employing the linearised adjacent equilibrium criterion. The resulted stability equations are solved via a hybrid Fourier-GDQ method. Some parametric studies are presented to study the influence of shell-foundation interaction. It is shown that elastic foundation enhances the thermal stability of shell and alters the buckling pattern of the shell. Furthermore, with the establishment of temperature dependency of the constituents for cylindrical shells under heat conduction, critical buckling temperatures are underestimated.

2. Equilibrium equations

Consider a cylindrical shell made of FGMs of thickness h, length L and mean radius R, referred to the coordinate system (x,θ,z), as shown in Fig. 1. The shell is made of two distinct constituents. Material non-homogeneous properties of a functionally graded material shell may be obtained by means of a proper homogenization method. The Voigt rule of mixture is frequently used for this reason. Furthermore, variation of the ceramic constituent should be known across the thickness. A simple power law function across the shell thickness is the commonly used type of property dispersion, see Shen (2004, 2007). Thus the material non-homogeneous properties of the FGM shell P, as a function of thickness coordinate, become

$$P(z,T) = P_i(T) + [P_o(T) - P_i(T)]\Big(\frac{1}{2} + \frac{z}{h}\Big)^k \tag{2.1}$$

where P_o and P_i are the corresponding properties of the outer and inner surfaces of the shell, respectively, and k is the power law index which takes the value greater than or equal to zero. In the present work, we assume that the elasticity modulus E, thermal conductivity K, and thermal expansion coefficient α, are described by Eq. (2.1). However, similarly to Shen (2004, 2007) and Bagherizadeh *et al.* (2012), Poisson's ratio ν is assumed to be constant with respect to temperature and thickness coordinate since it varies in a small range.

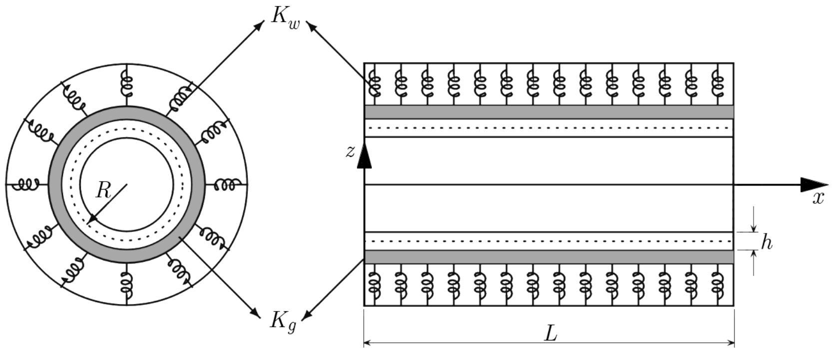

Fig. 1. Coordinate system and geometry of a cylindrical shell embedded in an elastic medium

The reported results of Shen (2004, 2007, 2013) reveal that for cylindrical shells in a moderately thick class of thickness, the buckling phenomenon may not occur. Therefore, in this study, we only focus on the shells that are thin enough to obey the conditions of classical shell theory. Based on the von-Karman assumptions and Donnell shell theory, the non-linear equilibrium equations are

$$
\begin{split}
&N_{xx,x}+\frac{1}{R}N_{x\theta,\theta}=0 \qquad\qquad N_{x\theta,x}+\frac{1}{R}N_{\theta\theta,\theta}=0\\
&M_{xx,xx}+\frac{2}{R}M_{x\theta,x\theta}+\frac{1}{R^2}M_{\theta\theta,\theta\theta}-\frac{1}{R}N_{\theta\theta}+N_{xx}w_{,xx}+\frac{1}{R^2}N_{\theta\theta}w_{,\theta\theta}\\
&\qquad+\frac{2}{R}N_{x\theta}w_{,x\theta}-K_w w+K_g\Big(w_{,xx}+\frac{1}{R^2}w_{,\theta\theta}\Big)=0
\end{split}
\tag{2.2}
$$

where in Eqs. (2.2) the stress resultants are defined as

$$
(N_{xx},N_{\theta\theta},N_{x\theta},M_{xx},M_{\theta\theta},M_{x\theta})=\int\limits_{-h/2}^{h/2}(\sigma_{xx},\sigma_{\theta\theta},\tau_{x\theta},z\sigma_{xx},z\sigma_{\theta\theta},z\tau_{x\theta})\,dz \tag{2.3}
$$

Definition of stress resultants in terms of mid-plane displacement components are obtained as

$$
\left\{\begin{array}{c}N_{xx}\\N_{\theta\theta}\\N_{x\theta}\\M_{xx}\\M_{\theta\theta}\\M_{x\theta}\end{array}\right\}=
\begin{bmatrix}
A_{11}&A_{12}&0&B_{11}&B_{12}&0\\
A_{21}&A_{22}&0&B_{12}&B_{22}&0\\
0&0&A_{66}&0&0&B_{66}\\
B_{11}&B_{12}&0&D_{11}&D_{12}&0\\
B_{21}&B_{22}&0&D_{12}&D_{22}&0\\
0&0&B_{66}&0&0&D_{66}
\end{bmatrix}
\left\{\begin{array}{c}
u_{,x}+\frac{1}{2}w_{,x}^2\\
\frac{1}{R}(v_{,\theta}+w)+\frac{1}{2R^2}w_{,\theta}^2\\
\frac{1}{R}u_{,\theta}+v_{,x}+\frac{1}{R}w_{,x}w_{,\theta}\\
-w_{,xx}\\
-\frac{1}{R^2}w_{,\theta\theta}\\
-\frac{2}{R}w_{,x\theta}
\end{array}\right\}
-\left\{\begin{array}{c}N^T\\N^T\\0\\M^T\\M^T\\0\end{array}\right\} \tag{2.4}
$$

In the above equations, the constant coefficients A_{ij}, B_{ij} and D_{ij} indicate the stretching, bending-stretching and bending stiffnesses respectively, which are calculated by

$$
(A_{ij},B_{ij},D_{ij})=\int\limits_{-0.5h}^{+0.5h}(Q_{ij},zQ_{ij},z^2Q_{ij})\,dz \tag{2.5}
$$

Besides, N^T and M^T are the thermal force and thermal moment resultants which are given by

$$(N^T, M^T) = \int\limits_{-0.5h}^{+0.5h} (1, z)\frac{1}{1-\nu}E(z,T)\alpha(z,T)(T-T_0)\,dz \tag{2.6}$$

Two types of boundary conditions are used in this research, which are immovable simply supported and immovable clamped. For the clamped edge C and simply supported S, boundary conditions are

$$\begin{aligned} &C: \quad u = v = w = w_{,x} = 0 \\ &S: \quad u = v = w = M_{xx} = 0 \end{aligned} \tag{2.7}$$

3. Temperature profile

As stated earlier, the temperature profile is assumed to be through the thickness direction only. Such an assumption is compatible with the design requirements of FGM shells. The one-dimensional heat conduction equation in the absence of heat generation for an FGM shell becomes (Hetnarski and Eslami, 2009)

$$(K(z,T)T_{,z})_{,z} = 0 \tag{3.1}$$

The boundary conditions of the temperature profile associated to the above equation are

$$T(+0.5h) = T_o \qquad\qquad T(-0.5h) = T_i \tag{3.2}$$

Solution of temperature profile (3.1) along with Eq. (3.2) may be obtained with the aid of Tailor series expansion when the temperature dependency is not included, see Shen (2007) and Kiani and Eslami (2013a,b). However, with the assumption of temperature dependent material properties, various surfaces of the shell inherit various temperature levels and consequently various thermo-mechanical properties. Therefore, a suitable numerical procedure should be adopted to extract the temperature profile through the shell thickness. Here, the central finite difference scheme is used for such solution. Applying the central finite difference method to Eq. (3.2) one arrives at

$$K(z^i,T^i)\frac{T^{i+1}-2T^i+T^{i-1}}{\varDelta^2} + K_{,z}(z^i,T^i)\frac{T^{i+1}-T^{i-1}}{2\varDelta} = 0 \tag{3.3}$$

and for boundary conditions (3.2), one obtains

$$T^N = T_o \qquad\qquad T^1 = T_i \tag{3.4}$$

where $i \in \{2,\ldots,N-1\}$, $z^i = -0.5h + (i-1)\varDelta$ and $\varDelta = h/(N-1)$. Besides, T^i indicates the temperature at the surface $z = z^i$. In a compact form and after imposing boundary conditions (3.4), the matrix representation of Eq. (3.3) becomes

$$\mathbf{K}_T(\mathbf{T})\mathbf{T} = \mathbf{F}_T(\mathbf{T}) \tag{3.5}$$

Since the material properties are temperature dependent, the stiffness matrix $\mathbf{K}_T(\mathbf{T})$ of Eq. (3.5) is function of the nodal temperatures $\mathbf{T} = \{T^1, T^2, \ldots, T^N\}^{\mathrm{T}}$. Consequently, at each step of heating, an iterative procedure should be performed to extract the temperature profile of the shell based on the assumption of temperature dependent material properties.

4. Prebuckling analysis

In general, unlike in plates, due to the initial curvature in cylindrical shells at the onset of thermal loading deformation occurs. Therefore, bending deformations exist in both pre-buckling and post-buckling equilibrium paths of the shell. The existence of lateral deflection at the onset of loading results in a non-linear pre-buckling equilibrium path. However, in cylindrical shells, the primary equilibrium path is frequently assumed to be linear. Such an assumption is shown to be efficient enough. For instance, in linear thermoelastic stability analysis of FGM cylindrical shells with simply-supported edges, Bagherizadeh *et al.* (2012) reported that their results have at most 6 percent deviation from those reported by Shen (2004) based on nonlinear equilibrium analysis. In this study, we only focus on linear membrane buckling analysis which means that prebuckling forces are obtained neglecting the bending and geometrically nonlinear effects in the prebuckling state. As proved by Bagherizadeh *et al.* (2012) prebuckling characteristics are

$$\begin{aligned} & u^0 = 0 && v^0 = 0 && w^0 = \frac{R}{A_{22} + R^2 K_w} N^T \\ & N^0_{\theta\theta} = -\frac{R^2 K_w}{A_{22} + R^2 K_w} N^T && N^0_{x\theta} = 0 && N^0_{xx} = -\frac{A_{22} - A_{12} + R^2 K_w}{A_{22} + R^2 K_w} N^T \end{aligned} \tag{4.1}$$

Since the pre-buckling study of this research is limited to membrane analysis, the above-mentioned prebuckling deformation/forces may be used for arbitrary classes of out-of-plane edge supports.

5. Stability equations

Stability equations of a cylindrical shell may be obtained based on the adjacent equilibrium criterion. To this end, a perturbed equilibrium position from the prebuckling state is considered. The equilibrium position in the prebuckling state is prescribed with components (u^0, v^0, w^0) given in Eq. (4.1). With an arbitrary perturbation (u^1, v^1, w^1), the shell experiences a new equilibrium configuration adjacent to the primary one described with the displacement components $(u^0 + u^1, v^0 + v^1, w^0 + w^1)$. Accordingly, linear stress resultants in the adjacent configuration are established as the sum of stress resultants in the prebuckling state and the perturbed stress resultant generated due to the incremental displacement field. The stability equations are then concluded as follows

$$\begin{aligned} & N^1_{xx,x} + \frac{1}{R} N^1_{x\theta,\theta} = 0 \qquad N^1_{x\theta,x} + \frac{1}{R} N^1_{\theta\theta,\theta} = 0 \\ & M^1_{xx,xx} + \frac{2}{R} M^1_{x\theta,x\theta} + \frac{1}{R^2} M^1_{\theta\theta,\theta\theta} - \frac{1}{R} N^1_{\theta\theta} + N^0_{xx} w^1_{,xx} + \frac{1}{R^2} N^0_{\theta\theta} w^1_{,\theta\theta} \\ & \qquad + \frac{2}{R} N^0_{x\theta} w^1_{,x\theta} - K_w w^1 + K_g \Big(w^1_{,xx} + \frac{1}{R^2} w^1_{,\theta\theta} \Big) = 0 \end{aligned} \tag{5.1}$$

Considering the linearised form of the resultant-displacement equations, three stability equations in terms of the perturbed components (u^1, v^1, w^1) are

$$\begin{aligned} & A_{11} u^1_{,xx} + \frac{A_{66}}{R^2} u^1_{,\theta\theta} + \frac{A_{12} + A_{66}}{R} v^1_{,x\theta} + \frac{A_{12}}{R} w^1_{,x} - B_{11} w^1_{,xxx} - \frac{B_{12} + 2B_{66}}{R^2} w^1_{,x\theta\theta} = 0 \\ & \frac{A_{12} + A_{66}}{R} u^1_{,x\theta} + \frac{A_{22}}{R^2} v^1_{,\theta\theta} + A_{66} v^1_{,xx} + \frac{A_{22}}{R^2} w^1_{,\theta} - \frac{B_{12} + 2B_{66}}{R} w^1_{,xx\theta} - \frac{B_{22}}{R^3} w^1_{,\theta\theta\theta} = 0 \end{aligned}$$

$$
\begin{aligned}
&B_{11}u^1_{,xxx}+\frac{B_{12}+2B_{66}}{R^2}u^1_{,x\theta\theta}+\frac{B_{12}+2B_{66}}{R}v^1_{,xx\theta}+\frac{B_{22}}{R^3}v^1_{,\theta\theta\theta}+\frac{2B_{12}}{R}w^1_{,xx}\\
&\quad+\frac{2B_{22}}{R^3}w^1_{,\theta\theta}-D_{11}w^1_{,xxxx}-\frac{2D_{12}+4D_{66}}{R^2}w^1_{,xx\theta\theta}-\frac{D_{22}}{R^4}w^1_{,\theta\theta\theta\theta}\\
&\quad-\frac{A_{22}}{R^2}w^1-\frac{A_{12}}{R}u^1_{,x}-\frac{A_{22}}{R^2}v^1_{,\theta}-K_w w^1+K_g\Big(w^1_{,xx}+\frac{1}{R^2}w^1_{,\theta\theta}\Big)\\
&\quad+N^0_{xx}w^1_{,xx}+\frac{2}{R}N^0_{x\theta}w^1_{,x\theta}+\frac{1}{R^2}N^0_{\theta\theta}w^1_{,\theta\theta}=0
\end{aligned}
\tag{5.2}
$$

Considering the periodicity conditions of the displacement field with respect to the circumferential coordinate, the next type of separation of variables is consistent with the stability equations

$$
\begin{Bmatrix} u_1(x,\theta)\\ v_1(x,\theta)\\ w_1(x,\theta)\end{Bmatrix}=\begin{bmatrix}\sin(m\theta) & 0 & 0\\ 0 & \cos(m\theta) & 0\\ 0 & 0 & \sin(m\theta)\end{bmatrix}\begin{Bmatrix} U(x)\\ V(x)\\ W(x)\end{Bmatrix}
\tag{5.3}
$$

where m is the half-wave number in the circumferential direction, and the functions $U(x)$, $V(x)$ and $W(x)$ are still unknown. Substitution of Eq. (5.3) into Eq. (5.2) yields a system of equations in terms of $U(x)$, $V(x)$ and $W(x)$. Solution of the system is carried out with the aid of GDQ method. Details on the GDQ method are not given here and readers may refer to Wu and Liu (1999) and Shu (2000).

After applying the GDQ method to Eqs. (5.2) with the aid of Eq. (5.3) and prebuckling forces (4.1), one may reach at

$$
\begin{aligned}
&\sum_{j=1}^{N}\Big[\Big(A_{11}C^{(2)}_{ij}-m^2\frac{A_{66}}{R^2}C^{(0)}_{ij}\Big)u_j-\frac{m(A_{12}+A_{66})}{R}C^{(1)}_{ij}v_j\\
&\qquad+\Big(-B_{11}C^{(3)}_{ij}+m^2\frac{B_{12}+2B_{66}}{R^2}C^{(1)}_{ij}+\frac{A_{12}}{R}C^{(1)}_{ij}\Big)w_j\Big]=0\\
&\sum_{j=1}^{N}\Big[\frac{m(A_{12}+A_{66})}{R}C^{(1)}_{ij}u_j+\Big(A_{66}C^{(2)}_{ij}-\frac{m^2A_{12}}{R^2}C^{(0)}_{ij}\Big)v_j\\
&\qquad+\Big(\frac{m(2B_{66}+B_{12})}{R}C^{(3)}_{ij}+m\frac{A_{22}}{R^2}C^{(0)}_{ij}+\frac{m^3B_{22}}{R^3}C^{(1)}_{ij}\Big)w_j\Big]=0\\
&\sum_{j=1}^{N}\Big[\Big(B_{11}C^{(3)}_{ij}-\frac{A_{12}}{R}C^{(1)}_{ij}-m^2\frac{B_{12}+2B_{66}}{R^2}C^{(1)}_{ij}\Big)u_j\\
&\qquad+\Big(\frac{mA_{22}}{R^2}C^{(0)}_{ij}-\frac{m(B_{12}+2B_{66})}{R}C^{(2)}_{ij}+m^3\frac{B_{22}}{R^3}C^{(0)}_{ij}\Big)v_j\\
&\qquad+\Big(\frac{2B_{12}}{R}C^{(2)}_{ij}-\frac{A_{22}}{R^2}C^{(0)}_{ij}-\frac{2m^2B_{22}}{R^3}C^{(0)}_{ij}+\frac{2m^2(D_{12}+2D_{66})}{R^2}C^{(2)}_{ij}\\
&\qquad-D_{11}C^{(4)}_{ij}-\frac{m^4D_{22}}{R^4}C^{(0)}_{ij}+N^0_{xx}C^{(2)}_{ij}-m^2N^0_{\theta\theta}C^{(0)}_{ij}\Big)w_j\Big]=0
\end{aligned}
\tag{5.4}
$$

and for the boundary conditions

$$
\begin{aligned}
&C:\quad u_i=v_i=w_i=\sum_{j=1}^{N}C^{(1)}_{ij}w_j=0\\
&S:\quad u_i=v_i=w_i=\sum_{j=1}^{N}B_{11}C^{(1)}_{ij}u_j-\frac{mB_{12}}{R}C^{(0)}_{ij}v_j-D_{11}C^{(2)}_{ij}w_j=0
\end{aligned}
\tag{5.5}
$$

The system of equations (5.4) with a proper choice of boundary conditions (5.5) results in

$$
\left(\begin{bmatrix}\mathbf{K}^{UU}_E & \mathbf{K}^{UV}_E & \mathbf{K}^{UW}_E\\ \mathbf{K}^{VU}_E & \mathbf{K}^{VV}_E & \mathbf{K}^{VW}_E\\ \mathbf{K}^{WU}_E & \mathbf{K}^{WV}_E & \mathbf{K}^{WW}_E\end{bmatrix}-\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{K}^{WW}_G\end{bmatrix}\right)\begin{Bmatrix}\mathbf{U}\\ \mathbf{V}\\ \mathbf{W}\end{Bmatrix}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{Bmatrix}
\tag{5.6}
$$

where in the above equation $\mathbf{K}_E$ is the elastic stiffness matrix and $\mathbf{K}_G$ is the geometric stiffness matrix. The elements of the geometric stiffness matrix contain the unknown temperature parameter N^T, and the elements of the elastic stiffness matrix contain the unknown circumferential half-wave number m. When the material properties are temperature independent, the above equations should be solved regardless of the through-the-thickness temperature profile. However, with regard to temperature dependency, various surfaces of the shell inherit different temperatures and thermo-mechanical properties and, therefore, elastic stiffness of the structures is also unknown. When the shell is subjected to a uniform temperature rise, the iterative process of Bagherizadeh *et al.* (2012) may be used successively. However, in the case of heat conduction across the thickness the following developed procedure may be used:

1) Temperatures of the inner and outer surfaces of the shell are assumed to be known. In the case of heat conduction across the thickness, the metal rich surface is assumed to be at 300 K.
2) Heat conduction equation (3.5) should be solved iteratively. To this end, at first, the thermal conductivity of the constituents is evaluated at the reference temperature. The temperature at each nodal point through the shell thickness is obtained as the solution to Eq. (3.5).
3) Thermal conductivities of the constituents at each surface of the shell are obtained based on the temperature dependent profile at the temperature levels of step (2). The stiffness matrix associated to Eq. (3.5) is established again. Solution of the temperature profile (3.5) is carried out again to extract the second estimation of the temperature profile.
4) The iteration in step (3) is repeated. This step should be done to reach the desirable accuracy.
5) Material properties such as Young's modulus, thermal expansion coefficient and thermal conductivity should be obtained at each surface of the shell according to the converged nodal temperatures in step (4). The thermal force resultant is obtained according to Eq. (2.6). The elastic stiffness matrix of Eq. (5.6) may now be established.
6) Equation (5.6) should be solved as an eigenvalue problem to obtain the critical buckling thermal force N_{cr}^T. This procedure should be done for various assumed circumferential mode numbers. The minimum value of N^T among all the obtained by consideration of various mode numbers is N_{cr}^T.
7) The obtained thermal force from step (6) is compared with the one obtained in step (5). When these forces are equal, the buckling temperature is obtained as the one assumed in step (1). Otherwise, a new temperature for the inner surface of the shell should be assumed, and steps (1) to (6) should be repeated. Generally, when the thermal force of step (5) is lower than the one obtained in step (6), the temperature should be estimated higher, otherwise, a lower value for temperature should be assumed.

6. Results and discussion

The procedure outlined in the previous section is used herein to study the critical buckling temperature difference of ceramic-metal functionally graded cylindrical shells. In parametric studies, the ceramic constituent is assumed as Si_3N_4 whereas the metal phase is assumed as SUS304. Properties of the aforesaid constituents are highly temperature dependent where the dependency on temperature may be expressed in terms of the following higher order Touloukian representation

$$P(T) = P_0(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3) \tag{6.1}$$

In the above equation, the constants P_i's are unique to each property and the constituent. For Si_3N_4 and SUS304, these constants are provided in Table 1. The abbreviate *TD* belongs to the temperature dependent material properties, whereas *TID* indicates the case when the thermo-mechanical properties of the shell are evaluated at the reference temperature $T_0 = 300\,\mathrm{K}$.

Table 1. Temperature dependent coefficients for SUS304 and Si_3N_4 from Shen (2007)

Material	Properties	P_{-1}	P_0	P_1	P_2	P_3
SUS304	α [K^{-1}]	0	+12.33E-6	+8.086E-4	0	0
	E [Pa]	0	+201.04E+9	+3.079E-4	−6.534E-7	0
	K [W/mK]	0	+15.379	−1.264E-3	+2.092E-6	−7.233E-10
	ν	0	+0.28	0	0	0
Si_3N_4	α [K^{-1}]	0	+5.8723E-6	+9.095E-4	0	0
	E [Pa]	0	+348.43E+9	−3.07E-4	+2.16E-7	−8.946E-11
	K [W/mK]	0	+13.723	−1.032E-3	+5.466E-7	−7.876E-11
	ν	0	+0.28	0	0	0

Two types of FGM cylindrical shells may be defined. In the first case, the inner surface is metal rich and the outer surface is ceramic rich, whereas in the second case the inner surface is ceramic rich and outer surface is metal rich. The convention A/B indicates a shell which is A-rich at the inner surface and B-rich at the outer surface.

The shell in divided into 51 nodes after the examination of convergence of the GDQ method. The constants of the elastic foundation are normalised according to the following equations and are used in this Section

$$k_w = \frac{K_w R^4}{D_m} \qquad k_g = \frac{K_g R^2}{D_m} \tag{6.2}$$

where in the above equations, D_m stands for the flexural rigidity of the metal-rich shell at the reference temperature. Table 2 yields a comparison on critical buckling temperatures of a class of shells. The numerical results of this study are compared with those reported by Shen (2004) based on the two-step perturbation method. It is seen that good agreement is observed at the onset of comparison. The little divergence between the results is due to the accountancy of nonlinear pre-buckling deformation in the analysis of Shen (2004).

Table 2. T_{cr} [K] of SUS304/Si_3N_4 and Si_3N_4/SUS304 *S-S* FGM cylindrical shells subjected to a uniform temperature rise. The properties of the shell are $L = \sqrt{300Rh}$, $R/h = 400$. Only the temperature dependent case of material properties is considered

k	Si_3N_4/SUS304		SUS304/Si_3N_4	
	Shen (2007)	Present	Shen (2007)	Present
0.0	382.27	392.79	459.44	478.73
0.2	390.64	402.31	437.63	454.49
0.5	399.89	412.55	422.02	437.27
1.0	410.25	424.15	410.25	424.15
2.0	422.21	437.55	401.14	413.90
3.0	429.15	445.30	397.18	409.43
5.0	437.15	454.20	393.25	404.98

Figure 2 demonstrates the variation of the minimum temperature obtained with respect to each of the circumferential mode numbers. The case of a shell with both edges clamped is considered. It is seen that the minimum buckling temperature of the shell is associated with the wave number $m = 15$. Therefore, the critical buckling temperature is the one corresponding to $m = 15$ and other values may belong to higher buckling temperatures.

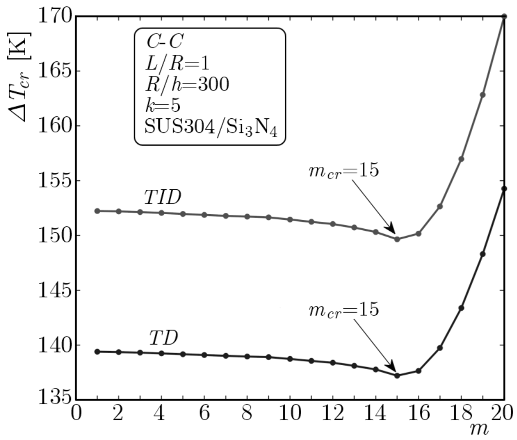

Fig. 2. Variation of the minimum buckling temperature of FGM shells subjected to a uniform temperature loading rise with respect to the circumferential mode number

To establish a simple example on the extraction of the critical buckling temperature in the heat conduction case, the variation of thermal forces according to the developed procedure are demonstrated in Fig. 3. It is seen that the critical buckling temperature difference in this case may be extracted as the intersection of two curves.

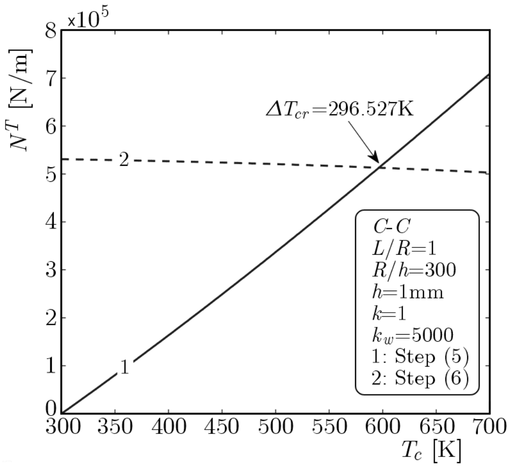

Fig. 3. Variations of thermal loads predicted by steps (5) and (6) with respect to the temperature rise parameter

Figure 4 depicts the influence of temperature dependency and composition rule on the critical buckling temperature difference of FGM cylindrical shells. The interaction between the shell and foundation is ignored. Two types of FGM shells are considered. It is seen that for both types, the *TD* case of material properties results in underestimation of the critical buckling temperature difference. The divergence of critical buckling temperature based on the *TID* assumption from the *TD* case is more observed at higher temperature levels. For Si_3N_4/SUS304, the critical buckling temperature difference increases with the increase of the power law index. In this type, the outer surface is metal rich. Therefore, when the power law index is equal to zero, the shell becomes fully metallic which is less stable in comparison to the ceramic shell. On the other hand, for SUS304/Si_3N_4, the critical buckling temperature difference decreases with the increase of the power law index. In this type, the outer surface is ceramic rich. Therefore, when the power law index is equal to zero, the shell becomes fully ceramic which is more stable and resistant to thermal bifurcation than the metal shell. In the case of a shell with $k = 1$, the critical buckling temperature difference for both SUS304/Si_3N_4 and Si_3N_4/SUS304 cases will be the same.

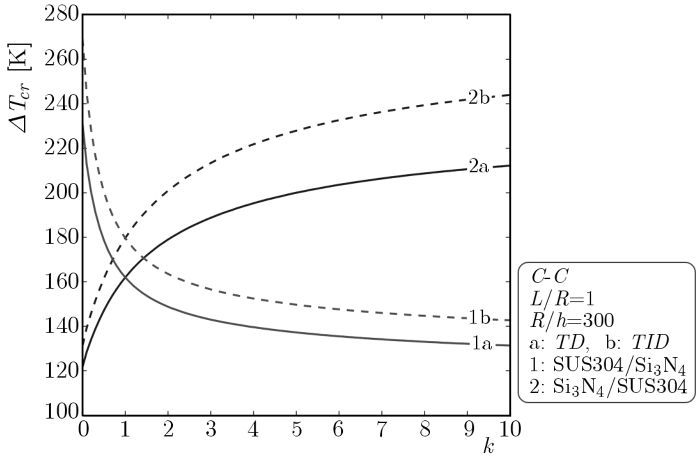

Fig. 4. Effect of the composition rule and temperature dependency on the critical buckling temperature difference of FGM shells subjected to a uniform temperature loading rise

Table 3 presents the critical buckling temperature difference, i.e. $T_i - T_o$ of an FGM cylindrical shell. The inner surface of the shell in ceramic-rich while the outer surface is metal rich. Therefore, the type of Si_3N_4/SUS304 cylindrical shell is considered. The outer surface of the shell is kept at the reference temperature. Various contact conditions are considered. It is seen that similar to the uniform temperature loading rise case, ΔT_{cr} [K] estimated by the *TID* assumption serves as the upper bound for the one obtained based on the *TD* assumption. Introduction of the Winkler elastic foundation does not change the buckling temperature of FGM cylindrical shells, significantly. However, the Winkler constant of the elastic foundation may increase or decrease the critical circumferential wave number. The shear constant of the Pasternak elastic foundation, however, affects both the critical buckling temperature difference and the buckling pattern of the shell. The power law index also may change the critical buckling pattern of the shell. As one may see from this table, the buckled configurations of shells with $k = 0, 1$ and 5 are totally different.

Table 3. ΔT_{cr} [K] for Si_3N_4/SUS304 FGM shells with *C-C* boundary conditions, $R/h = 300$, $L/R = 1$ subjected to heat conduction across the thickness. The number in parenthesis indicates the circumferential mode number

k		(k_w, k_g)				
		(0,0)	(1000,0)	(1000,200)	(5000,0)	(5000,200)
0	*TID*	262.245 (15)	262.412 (15)	293.123 (1)	263.004 (15)	293.510 (1)
	TD	238.127 (15)	238.451 (15)	264.125 (1)	239.102 (15)	264.761 (1)
0.5	*TID*	307.716 (15)	308.124 (15)	338.514 (1)	309.514 (15)	339.102 (1)
	TD	274.692 (15)	274.923 (15)	300.125 (1)	275.351 (15)	300.605 (1)
1	*TID*	333.708 (13)	334.116 (15)	365.120 (1)	335.108 (15)	366.108 (1)
	TD	295.476 (16)	295.980 (15)	320.184 (1)	296.527 (15)	320.910 (1)
2	*TID*	368.850 (16)	369.109 (16)	402.110 (1)	370.254 (15)	402.907 (1)
	TD	322.057 (16)	323.614 (16)	349.120 (1)	324.504 (15)	349.971 (1)
5	*TID*	424.442 (16)	424.699 (16)	461.752 (1)	426.124 (15)	462.527 (1)
	TD	369.451 (16)	369.792 (16)	398.051 (1)	370.527 (15)	398.758 (1)
10	*TID*	464.054 (16)	464.455 (16)	504.815 (1)	466.128 (15)	505.521 (1)
	TD	405.817 (16)	406.025 (16)	436.125 (1)	406.977 (15)	436.817 (1)

Table 4 presents the critical buckling temperature difference of FGM cylindrical shells for various values of the R/h ratio and contact conditions. As seen, the critical buckling temperature difference of the cylindrical shell decreases permanently with an increase in the R/h ratio. In fact, increasing the R/h ratio results in a lower flexural rigidity of the shell and, consequently, lower buckling temperature. It is observed that the higher the R/h ratio, the higher the critical circumferential mode number. The influence of the Winkler constant of the elastic foundation is almost negligible, whereas the shear layer of the Pasternak foundation enhances the critical buckling temperatures of the shell significantly. Numerical results accept that the effect of the power law index may be compensated with the shear effect of the Pasternak foundation.

Table 4. ΔT_{cr} [K] for the Si_3N_4/SUS304 FGM shells with *C-C* boundary conditions, $k = 1$ and $L/R = 1$ subjected to heat conduction across the thickness. The number in parenthesis indicates the circumferential mode number

R/h		(k_w, k_g)				
		(0,0)	(1000,0)	(1000,200)	(5000,0)	(5000,200)
100	*TID*	1000.807 (7)	1017.216 (9)	1294.645 (1)	1046.238 (8)	1296.510 (1)
	TD	740.125 (7)	751.315 (9)	956.122 (1)	770.751 (8)	956.264 (1)
200	*TID*	500.871 (11)	503.647 (13)	573.812 (1)	507.128 (11)	573.993 (1)
	TD	419.527 (11)	421.398 (13)	473.628 (1)	424.008 (11)	473.905 (1)
300	*TID*	333.708 (13)	334.116 (15)	365.120 (1)	335.108 (15)	366.108 (1)
	TD	295.476 (16)	295.980 (15)	320.184 (1)	296.527 (15)	320.910 (1)
400	*TID*	249.614 (18)	250.259 (16)	268.126 (1)	250.907 (18)	268.648 (1)
	TD	227.412 (18)	227.853 (16)	242.500 (1)	229.618 (18)	242.826 (1)
500	*TID*	199.517 (20)	200.007 (20)	211.405 (1)	200.540 (20)	211.826 (1)
	TD	185.216 (20)	185.409 (20)	195.124 (1)	185.927 (20)	196.758 (1)
600	*TID*	166.267 (22)	166.390 (22)	174.505 (1)	166.806 (22)	174.901 (1)
	TD	156.708 (22)	156.901 (22)	163.003 (1)	157.208 (22)	163.508 (1)

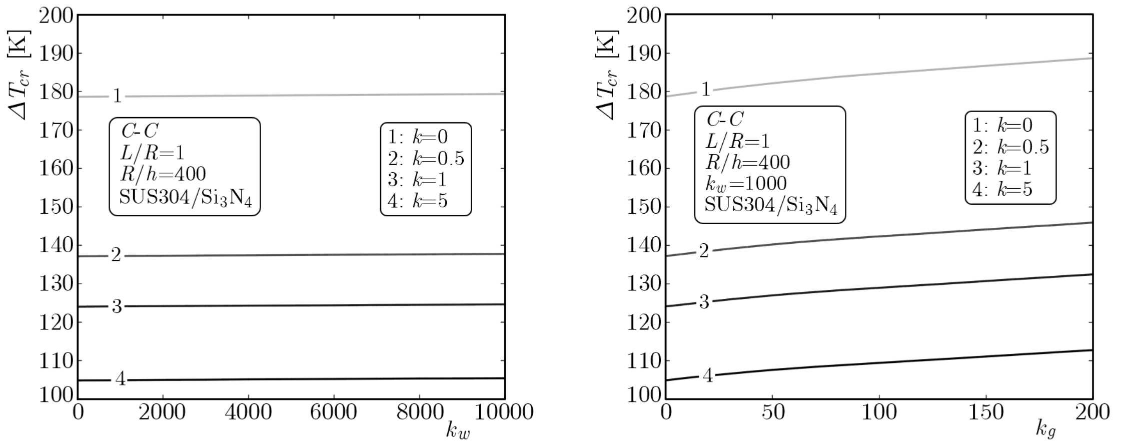

Fig. 5. Influence of the Winkler and Pasternak constants of the elastic foundation on the critical buckling temperature difference of *TD* FGM shells subjected to a uniform temperature loading rise

Figure 5 illustrates the variation of the critical buckling temperature difference of FGM shells in contact with the Winkler elastic foundation. Only the case of a shell with temperature dependent material properties is analysed. For such a condition, the shear constant of the elastic foundation should be equal to zero. A cylindrical shell made of SUS304/Si_3N_4 is considered. A wide range of the Winkler stiffness is considered. It is seen that the influence of the Winkler constant of the elastic foundation on ΔT_{cr} is almost negligible. This trend was also reported by Bagherizadeh *et al.* (2012) and is observed through the numerical results of Shen (2013). As expected, increasing the power law index results in a higher metal volume fraction and, consequently, decreases the critical buckling temperature difference.

The influence of the shear constant of the elastic foundation on the critical buckling temperature difference of FGM cylindrical shells is demonstrated in Fig. 5. Material properties of the shell are assumed to be temperature dependent. A wide range of the Pasternak constant of the elastic foundation is considered. It is seen that with the introduction of the Pasternak elastic foundation, the critical buckling temperature increases. The stiffer the foundation, the higher the buckling temperature.

7. Conclusion

Nonlinear equilibrium equations of FGM shells in contact with the Pasternak elastic foundation are obtained by means of the statical version of the virtual displacements principle. Donnell's type of kinematic relations combined with von-Karman's type of geometrical non-linearity are adopted as the basic assumptions. The pre-buckling deformations of the shell are obtained based on the linear membrane pre-buckling approach. The adjacent equilibrium criterion is used to extract the stability equations. Based on the GDQ method along the shell length combined with the Navier solution through the circumferential coordinate, the stability equations are solved as an eigenvalue problem, successively. It is shown that the Pasternak elastic foundation may enhance the buckling resistance of the shell. Furthermore, the influence of the Winkler constant of the elastic foundation is almost negligible, however, the shear layer may effectively delay the bifurcation. Both Winkler and shear layers of the elastic foundation may change the critical circumferential mode number. Furthermore, for both heat conduction and uniform temperature loadings, the regarding of the temperature dependency results in lower values of the critical buckling temperature difference. The divergence of the TD and TID cases is more revealed at higher temperature levels.

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