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SHORT RESEARCH COMMUNICATION

UNIQUENESS IN INCLUSION PROBLEMS WITH IMPERFECT INTERFACE

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We consider a non-standard boundary value problem characterizing deformations of a composite consisting of a arbitrarily-shaped elastic inclusion embedded in an infinite elastic matrix subjected to uniform remote stresses. The interface between the inclusion and the surrounding matrix is taken to be imperfect with 'spring-like' interface parameters describing the properties of the interface layer. We show that any classical solution to the boundary value problem is necessarily unique despite the fact that the asymptotic behaviour of the solution is not accommodated by the corresponding classical results from the same theory of elasticity.

Keywords: imperfect interface, inclusion, plane elasticity, uniqueness of solution

1. Introduction

Problems in composite mechanics involving elastic inclusions (inhomogeneities) embedded in an infinite matrix of a separate elastic material have attracted considerable attention from both theoreticians and practitioners alike. Earlier studies in this area focused on the assumption that the inclusion was perfectly bonded to the surrounding matrix (tractions and displacements were assumed to be continuous across the inclusion-matrix interface: see, for example, Eshelby, 1957, 1959). Recently, in an attempt to understand interface damage and its subsequent effect on the effective properties of composites, there has been increased emphasis on a class of problems in which the inclusion is *imperfectly* bonded to the matrix. Analyses in this context have been performed for linear isotropic materials (see, for example, Hashin, 1991a; Gao, 1995; Ru and Schiavone, 1997; Sudak *et al.*, 1999, and the references contained therein), linear anisotropic materials (for example, Ting and Schiavone, 2010) and for a class of materials undergoing finite plane deformations (for example, Wang, 2012). In many of these cases, the (imperfect) interface model used is the well-known 'spring-layer model' (Hashin, 1991b) in which tractions are continuous but displacements are discontinuous across the interface with 'jumps' in the displacement components assumed to be proportional to their corresponding interface traction components. This model is particularly attractive in that it can be used to account for micro-voids and micro--cracks almost always present in the interfacial region (Fan and Sze, 2001). Unfortunately, the model is almost always associated with a non-standard 'transmission-type' boundary value problem and although certain solutions for specific inclusion geometries and loading conditions have recently been established (see, for example, Ru and Schiavone, 1997; Sudak *et al.*, 1999; Ting and Schiavone, 2010) no rigorous analysis of the well-posedness of the corresponding boundary value problems has been undertaken. The question of the uniqueness of solution is particularly important since it forms the basis of most constructive methods (numerical or otherwise) currently being used to solve this important class of problems. Unfortunately, uniqueness theorems for this type of problem are not accommodated by those available for classical boundary value problems, essentially because of the peculiar nature of the boundary condition across the interface. In this paper, we establish uniqueness results for the problem describing the plane-strain deformations of an inclusion-matrix composite subjected to plane-strain deformations.

2. Formulation

Consider a domain in $\mathbb{R}^2$, infinite in extent, containing a single internal elastic inhomogeneity with elastic properties different than those of the surrounding matrix. The matrix and inhomogeneity occupy regions denoted by $S^{(1)}$ and $S^{(2)}$, respectively, while the inclusion-matrix interface is denoted by the smooth curve Γ. In what follows, the superscripts (1), (2) refer, respectively, to quantities corresponding to the regions $S^{(1)}$ and $S^{(2)}$.

The classical boundary-value problem describing the deformation of the composite in the absence of body forces is given by Hashin (1991b) and Constanda (1995):

Find a regular solution pair $\{u^{(1)}, u^{(2)}\}$ such that $u^{(2)} \in C^2(S^{(2)}) \cap C^1(\overline{S}^{(2)})$ and $u^{(1)} \in C^2(S^{(1)}) \cap C^1(\overline{S}^{(1)}) \cap \mathcal{A}$ and such that

$$\begin{aligned} &L^{(2)}u^{(2)} = 0 \qquad \text{in} \quad S^{(2)} \\ &L^{(1)}u^{(1)} = 0 \qquad \text{in} \quad S^{(1)} \\ &T^{(1)}u^{(1)} = T^{(2)}u^{(2)} \qquad \text{on} \quad \Gamma \\ &M(u^{(1)} - u^{(2)}) = T^{(2)}u^{(2)} + Mu^{(0)} \qquad \text{on} \quad \Gamma \end{aligned} \tag{2.1}$$

Here, u is the (2×1)-matrix describing the displacement field; L, T are the (2×2)-matrices of operators representing the governing equations and stress operators, respectively, of the plane--strain (see Constanda, 1995); M is the matrix $\begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}$ of (variable) interface parameters $m, n > 0$ prescribed on Γ (Hashin, 1991b; Sudak *et al.*, 1999); $u^{(0)}$ represents an additional displacement induced within the inhomogeneity by a uniform eigenstrain, and $\mathcal{A}$ represents the collection of displacement fields with asymptotic behaviour described by

$$\begin{aligned} &u^{(1)}(x,y) = (c_1x - c_2y + o(1), c_3x - c_4y + o(1)) \\ &\text{for} \quad (x,y) \in \mathbb{R}^2 \quad \text{as} \quad x^2 + y^2 \to \infty \end{aligned} \tag{2.2}$$

with known constants c_i, $i = 1, 2, 3, 4$. We note that Eq. (2.2) corresponds to the case of prescribed uniform remote stress and, as such, is not accommodated by the classical results of Constanda (1995).

3. Uniqueness Result

Consider the difference pair $\{w^{(1)}, w^{(2)}\}$ of any two solutions to boundary value problem (2.1)--(2.2). Let K_R be a circle with smooth boundary ∂K_R and radius R sufficiently large so that Γ lies inside K_R. Writing the Betti formulae (Constanda, 1995) for $\{w^{(1)}, w^{(2)}\}$ in the bounded regions $S^{(1)} \cap K_R$ and $S^{(2)}$, making use of the homogeneous boundary conditions for $\{w^{(1)}, w^{(2)}\}$ and asymptotic condition from (2.2) on $w^{(1)}$, we arrive at the equation

$$\int\limits_{S^{(2)}} E(w^{(2)}, w^{(2)})\, dA + \int\limits_{S^{(1)}} E(w^{(1)}, w^{(1)})\, dA + \int\limits_{\Gamma} \{M^{-1}Tw^{(2)}\}^{\mathrm{T}} Tw^{(2)}\, dS = 0$$

where E is the internal energy density. Given that $m, n > 0$ on Γ, we deduce that $w^{(1)}$, $w^{(2)}$ are arbitrary rigid displacements in each of $S^{(1)}$ and $S^{(2)}$, respectively. From the asymptotic conditions on $w^{(1)}$, we easily deduce that $w^{(1)} = 0$ in $\overline{S}^{(1)}$; hence $w^{(1)} = T^{(1)}w^{(1)} = 0$ on Γ. Boundary conditions $(2.1)_{2,3}$ then imply that $-Mw^{(2)} = 0$ on Γ, which, since M is invertible, means that $w^{(2)} = 0$ on Γ. Since it has already been determined that $w^{(2)}$ is an arbitrary rigid displacement in $S^{(2)}$, the continuity requirements of boundary value problem (2.1)-(2.2) mean that $w^{(2)} = 0$ in $\overline{S}^{(2)}$. Consequently, boundary value problem (2.1)-(2.2) has at most one regular solution pair $\{u^{(1)}, u^{(2)}\}$ which establishes the required uniqueness result.

It should be noted that particular solutions to (2.1)-(2.2) have been constructed by Sudak *et al.* (1999). The formal analysis of the existence of solutions to (2.1)-(2.2) using the boundary integral equation method will be the subject of a future paper. The same technique as that used above, with very little change in detail, can be applied to prove the uniqueness result for the corresponding problem of anti-plane shear deformations studied by Ru and Schiavone (1997) and Ting and Schiavone (2010).

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