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ON THE TURBULENT BOUNDARY LAYER OF A DRY GRANULAR AVALANCHE DOWN AN INCLINE. I. THERMODYNAMIC ANALYSIS

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Characteristics of the turbulent boundary and passive layers of an isothermal dry granular avalanche with incompressible grains are studied by the proposed zero-order turbulence closure model. The first and second laws of thermodynamics are applied to derive the equilibrium closure relations satisfying turbulence realizability conditions, with the dynamic responses postulated within a quasi-static theory. The established closure model is applied to analyses of a gravity-driven stationary avalanche down an incline to illustrate the distributions of the mean solid content, mean velocity, turbulent kinetic energy and dissipation across the flow layer, and to show the influence of turbulent fluctuation on the mean flow features compared with laminar flow solutions. In this paper, detailed thermodynamic analysis and equilibrium closure relations are summarized, with the dynamic responses, the complete closure model and numerical simulations reported in the second part.

Keywords: closure model, dry granular avalanche, thermodynamics, turbulence

1. Introduction

Dry granular systems are collections of a large amount of dispersive solid particles with interstices filled with a gas. When in motion, the interactions among the solid grains result from short--term instantaneous elastic and inelastic collisions and long-term enduring frictional contact and sliding. They play a structural role in the macroscopic behavior, and are characterized as the *microstructural effect* (Aranson and Tsimring, 2009; Ausloos *et al.*, 2005; GDR MiDi, 2004; Mehta, 2007; Pöschel and Brilliantiv, 2013; Rao and Nott, 2008). Depending on dominant microstructural grain-grain interactions, the quasi-static, dense and collisional flow states are defined, by which the avalanche is classified as a flow with rapid speed (flow in collisional state) (Pudasaini and Hutter, 2007; Pöschel and Brilliantiv, 2013; Rao and Nott, 2008). The microstructural effect significantly depends on flow speed, leading to a dry granular avalanche exhibiting distinct rheological characteristics (Oswald, 2009; Pudasaini and Hutter, 2007).

A dry granular flow experiences fluctuations on its macroscopic quantities, a phenomenon similar to turbulent motion of Newtonian fluids in three perspectives: (i) it results from two-fold grain-grain interactions, in contrast to that of Newtonian fluids induced by incoming flow instability, instability in the transition region or flow geometry (Batchelor, 1993; Tsinober, 2009); (ii) it emerges equally at slow speed in contrast to that of Newtonian fluids, which is dependent significantly on flow velocity, characterized by the critical Reynolds number; and (iii) while turbulent fluctuation induces most energy production with anisotropic eddies and energy dissipation with fairly isotropic eddies at the scales similar to the integral and Kolmogorov length scales in Newtonian fluids, respectively. Granular eddies at the inertia sub-range or Taylor microeddies are barely recognized. These imply that a dry granular flow can be considered a rheological fluid continuum with significant kinetic energy dissipation. Turbulent fluctuation induces energy cascades from the stress power at the mean scale toward the thermal dissipation at the subsequent length and time scales (Pudasaini and Hutter, 2007; Rao and Nott, 2008).

Field observations suggest that turbulent intensity is responsible for the entrainment of geophysical mass transportation, and a dry granular avalanche is conjectured to consist of two distinct layers: a very thin *turbulent boundary layer* immediately above the base, and a relatively thick *passive layer* above the former (Pudasaini and Hutter, 2007; Wang and Hutter, 2001). In the turbulent boundary layer, the grains collide intensively one with another, resulting in reduced base friction so that the avalanche can travel unexpected long distance. On the contrary, the dominant grain-grain interaction in the passive layer is a long-term one, causing the grains to behave as a lump solid.

To take into account the effect of turbulent fluctuation on mean flow features, a conventional Reynolds-filter process is applied to decompose the variables into the mean and fluctuating parts to obtain the balance equations of the mean fields with ergodic terms. They need be prescribed as functions of the mean fields, known as the *turbulent closure relations*, to arrive at a mathematically likely well-posed problem. By different prescriptions of the closure relations, turbulence closure models of different orders can be established (Batchelor, 1993).

Studies on turbulent characteristics of dry granular systems are so far yet complete. Various models for slow creeping and dense laminar flows (e.g. Faccanoni and Mangeney, 2013; Fang, 2009a, 2010; Jop, 2008; Jop *et al.*, 2006; Kirchner, 2002) and for rapid laminar flows (e.g. Campell, 2005; Dainel *et al.*, 2007; Fang, 2008a, 2008b; Savage, 1993; Wang and Hutter, 1999) haven been developed. The turbulent models by Ahmadi (1985), Ahmadi and Shahinpoor (1983) and Ma and Ahmadi (1985) used Prandtl's mixing length to account for the turbulent viscosity, with applications to simple shear flows. Although the influence of velocity fluctuation on the linear momentum balance was accounted for by Reynolds' stress, the fluctuating kinetic energy was not taken into account. Effort has been made to account for the fluctuating kinetic energy by the granular temperature (e.g. Goldhrisch, 2008; Luca *et al.*, 2004; Vescoci *et al.*, 2013; Wang and Hutter, 2001). Only the equilibrium closure relations were obtained; numerical simulations of Benchmark problems compared with experimental outcomes were insufficient; the conjecture of a two-layered avalanche remains unverified systematically.

Recently, the granular coldness, a similar concept to the granular temperature, has been extended to account for the influence of weak turbulent fluctuation induced by two-fold grain--grain interactions (Fang, 2016a; Fang and Wu, 2014a). A kinematic equation was used to describe the time evolution of the turbulent kinetic energy, with the turbulent dissipation considered a closure relation or an independent field resulting respectively in zero- and first-order turbulence closure models. While the mean porosity and velocity coincided to the experimental outcomes, the turbulent dissipation was shown to be similar to that of conventional Newtonian fluids in turbulent boundary layer flows. Although the first-order model was able to account for the influence of turbulent eddy evolution, the zero-order model was sufficient to capture the turbulent kinetic energy and dissipation distributions (Fang and Wu, 2014b).

Thus, the goal of the study is to establish a zero-order closure model for isothermal dry granular avalanches with incompressible grains. The first and second laws of thermodynamics, specifically the Müller-Liu entropy principle, are used to derive the equilibrium closure relations with the dynamic responses postulated within a quasi-static theory. The established model is applied to analyses of a gravity-driven stationary avalanche down an incline, compared with laminar flow solutions to illustrate the distributions of turbulent kinetic energy and dissipation with their influence on the mean flow features, and to verify the characteristics of the turbulent boundary and passive layers with their similarities to those of Newtonian fluids. The study is divided into two parts. Analysis of the Müller-Liu entropy principle and derived equilibrium closure relations are summarized in this paper, with the complete closure model and numerical simulations reported in the second part (Fang, 2016b). The mean balance equations and turbulent state space are given in Section 2, followed by the thermodynamic analysis in Section 3.

The derived equilibrium closure relations are summarized in Section 4, with the work concluded in Section 5.

2. Mean balance equations and turbulent state space

2.1. Mean balance equations

Following the balance equations for laminar motion and the Reynolds-filter process, the mean balance equations for a turbulent flow are given by Batchelor (1993) and Fang (2009a, 2014a)[1]

$$0 = \dot{\bar{\gamma}}\bar{\nu} + \bar{\gamma}\dot{\bar{\nu}} + \bar{\gamma}\bar{\nu}\nabla\cdot\bar{\mathbf{v}} \tag{2.1}$$

$$bf0 = \bar{\gamma}\bar{\nu}\dot{\bar{\mathbf{v}}} - \operatorname{div}(\bar{\mathbf{t}} + \mathbf{R}) - \bar{\gamma}\bar{\nu}\bar{\mathbf{b}} \tag{2.2}$$

$$\mathbf{0} = \bar{\mathbf{t}} - \bar{\mathbf{t}}^{\mathrm{T}} \qquad \mathbf{0} = \mathbf{R} - \mathbf{R}^{\mathrm{T}} \qquad \mathbf{0} = \bar{\mathbf{Z}} - \bar{\mathbf{Z}}^{\mathrm{T}} \tag{2.3}$$

$$0 = \bar{\gamma}\bar{\nu}\dot{\bar{e}} - \bar{\mathbf{t}}\cdot\bar{\mathbf{D}} + \nabla\cdot(\bar{\mathbf{q}} + \mathbf{Q}) - \bar{\gamma}\bar{\nu}\varepsilon - \bar{\gamma}\bar{\nu}\bar{r} - \ell\bar{\mathbf{h}}\cdot\nabla\dot{\bar{\nu}} + \bar{\gamma}\bar{\nu}\bar{f}\ell\dot{\bar{\nu}} - \bar{\gamma}\bar{\nu}H \tag{2.4}$$

$$0 = \bar{\gamma}\bar{\nu}\dot{\bar{\eta}} + \nabla\cdot(\bar{\boldsymbol{\phi}} + \boldsymbol{\phi}') - \bar{\gamma}\bar{\nu}\bar{\sigma} - \bar{\pi} \tag{2.5}$$

$$0 = \bar{\gamma}\bar{\nu}\ell\ddot{\bar{\nu}} - \nabla\cdot(\bar{\mathbf{h}} + \mathbf{H}) - \bar{\gamma}\bar{\nu}\bar{f} \tag{2.6}$$

$$\mathbf{0} = \mathring{\bar{\mathbf{Z}}} - \bar{\mathbf{\Phi}} \qquad (\mathring{\bar{\mathbf{Z}}} \equiv \dot{\bar{\mathbf{Z}}} - [\bar{\mathbf{\Omega}}, \bar{\mathbf{Z}}]) \tag{2.7}$$

$$0 = \bar{\gamma}\bar{\nu}\dot{k} - \mathbf{R}\cdot\bar{\mathbf{D}} - \nabla\cdot\mathbf{K} + \bar{\gamma}\bar{\nu}\varepsilon \tag{2.8}$$

$$0 = \bar{\gamma}\bar{\nu}\dot{s} - \ell\mathbf{H}\cdot\nabla\dot{\bar{\nu}} - \nabla\cdot\mathbf{L} + \bar{\gamma}\bar{\nu}H \tag{2.9}$$

with the ergodic terms,

$$\begin{aligned}
&0 = R_{ij} + \bar{\gamma}\bar{\nu}\overline{v_i'v_j'} && 0 = H_j - \ell R_{ij}\frac{\partial\bar{\nu}}{\partial x_i} && 0 = \phi_j' - \bar{\gamma}\bar{\nu}\overline{\eta' v_j'}\\
&0 = Q_j - \bar{\gamma}\bar{\nu}\overline{e'v_j'} && 0 = \bar{\gamma}\bar{\nu}\varepsilon - \overline{t_{ij}'\frac{\partial v_i'}{\partial x_j}} && 0 = M_{ij} - \ell\overline{h_i'v_j'}\\
&0 = \bar{\gamma}\bar{\nu}k - \frac{1}{2}\bar{\gamma}\bar{\nu}\overline{v_i'v_j'} && 0 = K_j - \overline{t_{ij}'v_i'} - \frac{1}{2}R_{iij} &&\\
&0 = \bar{\gamma}\bar{\nu}s + \frac{1}{2}\ell^2 R_{ij}\frac{\partial\bar{\nu}}{\partial x_i}\frac{\partial\bar{\nu}}{\partial x_j} && 0 = \bar{\gamma}\bar{\nu}H - M_{ij}\frac{\partial^2\bar{\nu}}{\partial x_i x_j} - \bar{\gamma}\bar{\nu}d &&\\
&0 = R_{ijk} + \bar{\gamma}\bar{\nu}\overline{v_i'v_j'v_k'} && 0 = \bar{\gamma}\bar{\nu}d - \ell\Big(\overline{h_i'\frac{\partial v_j'}{\partial x_i}} - \bar{\gamma}\bar{\nu}\overline{f'v_j'}\Big)\frac{\partial\bar{\nu}}{\partial x_j} &&\\
&0 = L_j - M_{ji}\frac{\partial\bar{\nu}}{\partial x_i} - \frac{1}{2}\ell^2 R_{ijk}\frac{\partial\bar{\nu}}{\partial x_i}\frac{\partial\bar{\nu}}{\partial x_k} && &&
\end{aligned} \tag{2.10}$$

in which $\bar{\nu}$ is the mean volume fraction defined as the total mean solid content divided by the volume of a representative volume element (RVE), and $\mathbf{A}\cdot\mathbf{B} \equiv \operatorname{tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}}) = \operatorname{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B})$, $[\mathbf{A},\mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA}$ for two arbitrary second-rank tensors $\mathbf{A}$ and $\mathbf{B}$. Variables and parameters arising in (2.1)-(2.10) are defined in Table 1, with KE and div abbreviations of kinetic energy and divergence, respectively. Quantities in (2.10), with the others shown later, are classified as closure relations.

Equations (2.1)-(2.5) are respectively the conventional mean balances of mass, linear momentum, angular momentum, internal energy and entropy for a fluid continuum in turbulent motion with the symmetry of the mean Cauchy stress demanded and the mean density $\bar{\rho}$ decomposed into $\bar{\rho} = \bar{\gamma}\bar{\nu}$. This introduces $\bar{\nu}$, considered an internal variable, with its time evolution

[1]It is equally possible to conduct the study by using the Naghdi-Green approach with original Reynolds' treatment, see e.g. Bilicki and Badur (2003).

Table 1. Variables and parameters in the mean balance equations

$\bar{\mathbf{b}}$	mean specific body force	$\bar{\mathbf{D}}$	symmetric part of $\bar{\mathbf{v}} \otimes \nabla$
$\bar{e}$	mean specific internal energy	$\bar{f}$	mean production associated with $\ell\dot{\bar{\nu}}$
$\bar{\mathbf{h}}$	mean flux associated with $\ell\dot{\bar{\nu}}$	H	specific turbulent dissipation of $\bar{\gamma}\bar{\nu}s$
$\mathbf{H}$	turbulent flux associated with $\ell\dot{\bar{\nu}}$	k	specific turbulent KE
$\mathbf{K}$	flux associated with $\bar{\gamma}\bar{\nu}k$	ℓ	constant internal length
$\mathbf{L}$	flux associated with $\bar{\gamma}\bar{\nu}s$	$\bar{\mathbf{q}}$	mean heat flux
$\mathbf{Q}$	turbulent heat flux	$\bar{r}$	mean specific energy supply
$\mathbf{R}$	Reynolds' stress	s	specific turbulent configurational KE
T	transpose	$\bar{\mathbf{t}}$	mean Cauchy stress
$\bar{\mathbf{v}}$	mean velocity	$\bar{\mathbf{Z}}$	mean internal friction
α	arbitrary quantity	$\bar{\alpha}$	time-averaged value of α
α'	fluctuating value of α	$\dot{\alpha}$	material derivative of α with respect to $\bar{\mathbf{v}}$
$\bar{\gamma}$	mean true mass density of solid grains	ε	specific turbulent dissipation of $\bar{\gamma}\bar{\nu}k$
$\bar{\eta}$	mean specific entropy	$\bar{\nu}$	mean volume fraction
$\bar{\pi}$	mean entropy production	$\bar{\sigma}$	mean specific entropy supply
$\bar{\boldsymbol{\phi}}$	mean entropy flux	$\boldsymbol{\phi}'$	turbulent entropy flux
$\bar{\boldsymbol{\Omega}}$	mean orthogonal rotation of RVE	∇	Nabla operator

described by the revised Goodman-Cowin model (2.6) for rapid flows (Fang, 2009a). It describes a self-equilibrated stress system, with constant internal length ℓ denoting the characteristic length of the grains (Fang, 2008a, 2009a; Wang and Hutter, 1999). Since the grain arrangements are assumed to be independent of motion of the granular body, variation in $\ell\dot{\bar{\nu}}$ provides extra energies to the internal energy balance, as indicated by $(-\ell\bar{\mathbf{h}} \cdot \nabla\dot{\bar{\nu}} + \bar{\gamma}\bar{\nu}\bar{f}\ell\dot{\bar{\nu}})$ in (2.4).

To account for the rate-independent characteristics, an Euclidean frame-indifferent, symmetric second-rank tensor $\bar{\mathbf{Z}}$ is introduced as an internal variable for the mean internal frictional and other non-conservative forces inside a granular microcontinuum (Svendsen and Hutter, 1999; Svendsen *et al.*, 1999), with $\mathring{\bar{\mathbf{Z}}}$ denoting the Zaremba-Jaumann corotational derivative of $\bar{\mathbf{Z}}$. It is a phenomenological generalization of the Mohr-Coulomb model for a granular material at low energy and high-grain volume fraction (see e.g. Fang, 2009b), with its time evolution described kinematically by (2.7). The mean internal friction $\bar{\mathbf{Z}}$ with its time evolution (2.7) are widely used for laminar and turbulent formulations (e.g. Fang, 2008a, 2009a, 2010; Fang and Wu, 2014a; Kirchner, 2002) and provide the foundation of a dry granular heap at an equilibrium state, when the grains are incompressible (Fang, 2009a).

Equation (2.8) is the evolution of the *turbulent kinetic energy*, $\bar{\gamma}\bar{\nu}k$, derived by taking the inner product of the velocity with the balance of linear momentum followed by the Reynolds--filter process. It is considered to account for the influence of the energy cascade in turbulent flows (Batchelor, 1993; Pudasaini and Hutter, 2007; Rao and Nott, 2008). The turbulent kinetic energy is generated by Reynolds' stress through mean shearing at the mean scale, transferred subsequently via the flux $\mathbf{K}$ at various length and time scales, and eventually dissipated at the smallest scale (the Kolmogorov scale) by the *turbulent dissipation*, $\bar{\gamma}\bar{\nu}\varepsilon$. Similarly, the fluctuation of the characteristic velocity $\ell\dot{\nu}$ in (2.6) also contributes an additional turbulent kinetic energy, the *turbulent configurational kinetic energy*, $\bar{\gamma}\bar{\nu}s$, with its time evolution kinematically described by (2.9) with the flux $\mathbf{L}$ and *turbulent configurational dissipation*, $\bar{\gamma}\bar{\nu}H$ (Luca *et al.*, 2004). Thus, two turbulent kinetic energies appear: $\bar{\gamma}\bar{\nu}k$ for the turbulent fluctuation on the bulk material velocity, and $\bar{\gamma}\bar{\nu}s$ for that on the characteristic velocity of the grain configuration. Two-fold turbulent dissipations $\bar{\gamma}\bar{\nu}\varepsilon$ and $\bar{\gamma}\bar{\nu}H$ are considered closure relations, with their eventual thermal effects indicated in (2.4). Since the time evolutions of $\bar{\mathbf{Z}}$, k and s are described by using kinematic equations, their variations do not provide additional energy contributions in the internal energy balance equation.

2.2. Turbulent state space

With these, the quantities

$$
\begin{aligned}
\mathcal{P} &= \{\bar{\gamma}, \bar{\nu}, \bar{\mathbf{v}}, \bar{\mathbf{Z}}, \vartheta^M, \vartheta^T, \vartheta^G\} \\
\mathcal{C} &= \{\bar{\mathbf{t}}, \mathbf{R}, \bar{e}, \bar{\mathbf{q}}, \mathbf{Q}, \bar{\eta}, \boldsymbol{\phi}^T, \bar{\mathbf{h}}, \mathbf{H}, \bar{f}, \bar{\boldsymbol{\Phi}}, k, s, \mathbf{K}, \mathbf{L}, \varepsilon, H\}
\end{aligned}
\tag{2.11}
$$

are introduced respectively as the primitive mean fields and closure relations, by which $\mathcal{C}$ should be constructed based on the turbulent state space given by

$$
\mathcal{Q} = \{\nu_0, \bar{\nu}, \dot{\bar{\nu}}, \mathbf{g}_1, \bar{\gamma}, \mathbf{g}_2, \vartheta^M, \mathbf{g}_3, \vartheta^T, \mathbf{g}_4, \vartheta^G, \mathbf{g}_5, \bar{\mathbf{D}}, \bar{\mathbf{Z}}\} \qquad \mathcal{C} = \hat{\mathcal{C}}(\mathcal{Q}) \tag{2.12}
$$

with $\mathbf{g}_1 \equiv \nabla\bar{\nu}$, $\mathbf{g}_2 \equiv \nabla\bar{\gamma}$, $\mathbf{g}_3 \equiv \nabla\vartheta^M$, $\mathbf{g}_4 \equiv \nabla\vartheta^T$, $\mathbf{g}_5 \equiv \nabla\vartheta^G$ and $\boldsymbol{\phi}^T \equiv \bar{\boldsymbol{\phi}} + \boldsymbol{\phi}'$. In $(2.11)_1$, ϑ^M is the *material coldness*, which can be shown to be the inverse of an empirical material temperature θ^M for simple materials (Hutter and Jöhnk, 2004). The turbulent kinetic energy is expressed conventionally by the *granular temperature* θ^T (Goldhirsch, 2008; Vescovi *et al.*, 2013; Wang and Hutter, 2001) or alternatively by the *granular coldness* ϑ^T (Fang and Wu, 2014a; Luca *et al.*, 2004).[2] In a similar manner, the turbulent configurational kinetic energy is expressed by the *granular configurational temperature* θ^G, or alternatively by the *granular configurational coldness* ϑ^G (Luca *et al.*, 2004). Two-fold granular coldnesses are used as implicit indices of the variations in $\bar{\gamma}\bar{\nu}k$, $\bar{\gamma}\bar{\nu}s$ and their dissipations.

State space (2.12) is proposed based on Truesdell's equi-presence principle and principle of frame-indifference, with ν_0 the value of $\bar{\nu}$ in the reference configuration, included due to its influence on the behavior of flowing granular matter (Luca *et al.*, 2004). The quantities ν_0, $\bar{\nu}$, $\dot{\bar{\nu}}$, $\mathbf{g}_1$, $\bar{\gamma}$ and $\mathbf{g}_2$ are for the elastic effect, corresponding to $\bar{\rho}$, $\dot{\bar{\rho}}$ and $\nabla\bar{\rho}$ for complex rheological fluids; ϑ^M and $\mathbf{g}_3$ represent temperature-dependency of physical properties; ϑ^T and $\mathbf{g}_4$ stand for influence of turbulent kinetic energy and dissipation; ϑ^G and $\mathbf{g}_5$ denote the effect of turbulent configurational kinetic energy and dissipation; while $\bar{\mathbf{D}}$ and $\bar{\mathbf{Z}}$ are for viscous and rate-independent effects, respectively. Since $\{\bar{\mathbf{t}}, \bar{\mathbf{q}}, \bar{\mathbf{h}}, \bar{\boldsymbol{\Phi}}\}$ are objective, and $\mathbf{v}'$ (the fluctuating velocity) is also objective, the quantities $\{\mathbf{R}, \mathbf{Q}, \mathbf{H}, \mathbf{K}, \mathbf{L}\}$ are equally objective, with which (2.3) is fulfilled.

3. Thermodynamic analysis

3.1. Entropy inequality

Turbulence realizability conditions require that during a physically admissible process, the second law of thermodynamics with a local form of non-negative entropy production, and all balance equations should be satisfied simultaneously (Rung *et al.*, 1999; Sadiki and Hutter, 2000). This can be achieved by considering the mean balance equations as the constraints of inequality (2.5) via the method of Lagrange multipliers, viz.[3]

[2] The simple relation between θ^M and ϑ^M does not hold for θ^T and ϑ^T in dry granular systems. Only the relation of $\vartheta^T = \hat{\vartheta}^T(\theta^T, \dot{\theta}^T)$ is understood, which holds equally for ϑ^G and θ^G. In addition, the coldness idea is more physically meaningful when materials exhibit no memory effect. Since a dry granular matter in rapid motion exhibits a relatively insignificant memory effect, the coldness is used as the first approximation.

[3] We follow the Müller-Liu approach, from our own experience it can deliver more possible findings than the classical Coleman-Noll approach for granular media thermodynamics. However, the classical Coleman-Noll approach is widely used to conduct thermodynamic analyses, see e.g. Cieszko (1996), Kubik (1986), Schrefler *et al.* (2009), Sobieski (2009), Wilmański (1996).

$$\begin{aligned}\bar{\pi} = {}& \bar{\gamma}\bar{\nu}\dot{\bar{\eta}} + \nabla\cdot\boldsymbol{\phi}^T - \bar{\gamma}\bar{\nu}\bar{\sigma} - \lambda^{\bar{\gamma}}(\dot{\bar{\gamma}}\bar{\nu} + \bar{\gamma}\dot{\bar{\nu}} + \bar{\gamma}\bar{\nu}\nabla\cdot\bar{\mathbf{v}}) - \boldsymbol{\lambda}^{\bar{\mathbf{v}}}\cdot\Big(\bar{\gamma}\bar{\nu}\dot{\bar{\mathbf{v}}} - \operatorname{div}(\bar{\mathbf{t}} + \mathbf{R}) - \bar{\gamma}\bar{\nu}\bar{\mathbf{b}}\Big)\\ & - \lambda^{\bar{e}}\Big(\bar{\gamma}\bar{\nu}\dot{\bar{e}} - \bar{\mathbf{t}}\cdot\bar{\mathbf{D}} + \nabla\cdot(\bar{\mathbf{q}} + \mathbf{Q}) - \bar{\gamma}\bar{\nu}\varepsilon - \bar{\gamma}\bar{\nu}\bar{r} - \ell\bar{\mathbf{h}}\cdot\nabla\dot{\bar{\nu}} + \bar{\gamma}\bar{\nu}\bar{f}\ell\dot{\bar{\nu}} - \bar{\gamma}\bar{\nu}H\Big)\\ & - \lambda^{\bar{\nu}}\Big(\bar{\gamma}\bar{\nu}\ell\ddot{\bar{\nu}} - \nabla\cdot(\bar{\mathbf{h}} + \mathbf{H}) - \bar{\gamma}\bar{\nu}\bar{f}\Big) - \boldsymbol{\lambda}^{\bar{\mathbf{Z}}}\cdot(\dot{\bar{\mathbf{Z}}} - [\boldsymbol{\Omega}, \bar{\mathbf{Z}}] - \bar{\boldsymbol{\Phi}})\\ & - \lambda^k(\bar{\gamma}\bar{\nu}\dot{k} - \mathbf{R}\cdot\bar{\mathbf{D}} - \nabla\cdot\mathbf{K} + \bar{\gamma}\bar{\nu}\varepsilon) - \lambda^s(\bar{\gamma}\bar{\nu}\dot{s} - \ell\mathbf{H}\cdot\nabla\dot{\bar{\nu}} - \nabla\cdot\mathbf{L} + \bar{\gamma}\bar{\nu}Ht) \geqslant 0\end{aligned} \tag{3.1}$$

with $\lambda^{\bar{\gamma}}$, $\boldsymbol{\lambda}^{\bar{\mathbf{v}}}$, $\lambda^{\bar{e}}$, $\lambda^{\bar{\nu}}$, $\boldsymbol{\lambda}^{\bar{\mathbf{Z}}}$, λ^k and λ^s being the corresponding Lagrange multipliers. Since $\{\dot{\vartheta}^M, \dot{\vartheta}^T, \dot{\vartheta}^G\}$ are not considered in $(2.11)_1$, it is assumed that (Hutter and Jöhnk, 2004)

$$\begin{aligned}&\lambda^{\bar{e}} = \vartheta^M \qquad \lambda^k = \vartheta^T \qquad \lambda^s = \vartheta^G\\ &\vartheta^M\psi^T \equiv \vartheta^M\bar{e} + \vartheta^T k + \vartheta^G s - \bar{\eta}\end{aligned} \tag{3.2}$$

with ψ^T the specific *turbulent Helmholtz free energy*. Since material behavior is required to be independent of external supplies, it follows $(-\bar{\gamma}\bar{\nu}\bar{\sigma} + \bar{\gamma}\bar{\nu}\boldsymbol{\lambda}^{\bar{\mathbf{v}}}\cdot\bar{\mathbf{b}} + \vartheta^M\bar{\gamma}\bar{\nu}\bar{r}) = 0$, an equation determining the mean entropy supply. Substituting (2.11)-(2.12) and (3.2) into (3.1) yields

$$\bar{\pi} = \mathbf{a}\cdot\boldsymbol{\mathcal{X}} + b \geqslant 0 \tag{3.3}$$

with $\boldsymbol{\mathcal{X}} = \{\dot{\bar{\mathbf{v}}}, \ddot{\bar{\nu}}, \dot{\bar{\gamma}}, \dot{\vartheta}^M, \dot{\vartheta}^T, \dot{\vartheta}^G, \dot{\mathbf{g}}_2, \dot{\mathbf{g}}_3, \dot{\mathbf{g}}_4, \dot{\mathbf{g}}_5, \dot{\bar{\mathbf{D}}}, \dot{\bar{\mathbf{Z}}}, \bar{\boldsymbol{\Omega}}, \nabla\nu_0, \nabla\dot{\bar{\nu}}, \nabla\mathbf{g}_1, \nabla\mathbf{g}_2, \nabla\mathbf{g}_3, \nabla\mathbf{g}_4, \nabla\mathbf{g}_5, \nabla\bar{\mathbf{D}}, \nabla\bar{\mathbf{Z}}\}$, and $\dot{\mathbf{g}}_1 = \nabla\dot{\bar{\nu}} - \mathbf{g}_1\nabla\bar{\mathbf{v}}$ has been used. Since the vector $\boldsymbol{\mathcal{X}}$ is the set of time- and space-independent variations of $\mathcal{Q}$, and the vector $\mathbf{a}$ and scalar b are only functions of (2.12), (3.3) is linear in $\boldsymbol{\mathcal{X}}$. Since $\boldsymbol{\mathcal{X}}$ can take any values, it would be possible to violate (3.3) unless

$$\mathbf{a} = \mathbf{0} \qquad \text{and} \qquad b \geqslant 0 \tag{3.4}$$

Equation $(3.4)_1$ leads to

$$\psi^T_{,\alpha} = \mathbf{0} \qquad \alpha \in \{\dot{\bar{\nu}}, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4, \mathbf{g}_5, \bar{\mathbf{D}}\} \tag{3.5}$$

for the restrictions on ψ^T; and

$$\begin{aligned}&\boldsymbol{\lambda}^{\bar{\mathbf{v}}} = \mathbf{0} \qquad \lambda^{\bar{\gamma}} = -\bar{\gamma}\vartheta^M\psi^T_{,\bar{\gamma}} \qquad \boldsymbol{\lambda}^{\bar{\mathbf{Z}}} = -\bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\bar{\mathbf{Z}}}\\ &\boldsymbol{\lambda}^{\bar{\mathbf{Z}}}\bar{\mathbf{Z}} = \bar{\mathbf{Z}}\boldsymbol{\lambda}^{\bar{\mathbf{Z}}} \qquad \lambda^{\bar{\nu}} = -\vartheta^M\ell^{-1}\psi^T_{,\dot{\bar{\nu}}}\end{aligned} \tag{3.6}$$

for the Lagrange multipliers, determined with the prescribed ψ^T; and

$$\bar{e} = \psi^T + \vartheta^M\psi^T_{,\vartheta^M} \qquad k = \vartheta^M\psi^T_{,\vartheta^T} \qquad s = \vartheta^M\psi^T_{,\vartheta^G} \tag{3.7}$$

for three specific energies; and

$$\begin{aligned}&\mathbf{0} = \boldsymbol{\phi}^T_{,\nu_0} - \vartheta^M(\bar{\mathbf{q}} + \mathbf{Q})_{,\nu_0} + \vartheta^T\mathbf{K}_{,\nu_0} + \vartheta^G\mathbf{L}_{,\bar{\nu}_0}\\ &\mathbf{0} = \boldsymbol{\phi}^T_{,\dot{\bar{\nu}}} - \vartheta^M(\bar{\mathbf{q}} + \mathbf{Q})_{,\dot{\bar{\nu}}} + \vartheta^T\mathbf{K}_{,\dot{\bar{\nu}}} + \vartheta^G\mathbf{L}_{,\dot{\bar{\nu}}} + \ell(\vartheta^M\bar{\mathbf{h}} + \vartheta^G\mathbf{H}) - \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\mathbf{g}_1}\\ &\mathbf{0} = \boldsymbol{\phi}^T_{,\mathbf{A}} - \vartheta^M(\bar{\mathbf{q}} + \mathbf{Q})_{,\mathbf{A}} + \vartheta^T\mathbf{K}_{,\mathbf{A}} + \vartheta^G\mathbf{L}_{,\mathbf{A}}\\ &\mathbf{0} = \operatorname{sym}\Big(\boldsymbol{\phi}^T_{,\mathbf{g}} - \vartheta^M(\bar{\mathbf{q}} + \mathbf{Q})_{,\mathbf{g}} + \vartheta^T\mathbf{K}_{,\mathbf{g}} + \vartheta^G\mathbf{L}_{,\mathbf{g}}\Big)\end{aligned} \tag{3.8}$$

with $\mathbf{A} \in \{\bar{\mathbf{D}}, \bar{\mathbf{Z}}\}$ and $\mathbf{g} \in \{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4, \mathbf{g}_5\}$. In obtaining (3.8), $\psi^T \neq \hat{\psi}^T(\cdot, \dot{\bar{\nu}})$ has been assumed, with which $\lambda^{\bar{\nu}}$ vanishes. This is motivated by that ψ^T is influenced significantly by the variations in $\bar{\nu}$, but not its time rate of change (Fang, 2009a; Kirchner, 2002; Luca *et al.*, 2004; Pudasaini and Hutter, 2007; Wang and Hutter, 1999) and justified for most dry granular systems in the collisional state.

Equation $(3.4)_2$ yields the residual entropy inequality, viz.

$$\begin{aligned}\bar{\pi} = &\Big(-\bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\bar{\nu}} + \bar{\gamma}^2\vartheta^M\psi^T_{,\bar{\gamma}} - \bar{\gamma}\bar{\nu}\vartheta^M\bar{f}\ell\Big)\dot{\bar{\nu}} + \sum_g \Big(\boldsymbol{\phi}^T_{,g} - \vartheta^M(\bar{\mathbf{q}}+\mathbf{Q})_{,g} + \vartheta^T\mathbf{K}_{,g} \\ &+ \vartheta^G\mathbf{L}_{,g}\Big)\cdot\nabla g + \Big(\vartheta^M\bar{\mathbf{t}} + \vartheta^T\mathbf{R} + \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\mathbf{g}_1}\otimes\mathbf{g}_1 + \bar{\gamma}^2\bar{\nu}\vartheta^M\psi^T_{,\bar{\gamma}}\mathbf{I}\Big)\cdot\bar{\mathbf{D}} \\ &+ \bar{\gamma}\bar{\nu}\Big(\varepsilon(\vartheta^M - \vartheta^T) + H(\vartheta^M - \vartheta^G) - \vartheta^M\psi^T_{,\bar{\mathbf{Z}}}\cdot\bar{\boldsymbol{\Phi}}\Big) \geqslant 0\end{aligned} \tag{3.9}$$

with $\otimes$ denoting the dyadic product; $\mathbf{I}$ the second-rank identity tensor; and $g \in \{\bar{\nu}, \bar{\gamma}, \vartheta^M, \vartheta^T, \vartheta^G\}$.

3.2. Extra entropy flux

Define the extra entropy flux $\boldsymbol{\xi}$, viz.

$$\boldsymbol{\xi} \equiv \boldsymbol{\phi}^T - \vartheta^M(\bar{\mathbf{q}}+\mathbf{Q}) + \vartheta^T\mathbf{K} + \vartheta^G\mathbf{L} \tag{3.10}$$

Substituting (3.10) into $(3.8)_1$ and $(3.8)_{2,3}$, results respectively in

$$\mathbf{0} = \boldsymbol{\xi}_{,\nu_0} \qquad \mathbf{0} = \boldsymbol{\xi}_{,\mathbf{A}} \qquad \mathbf{0} = \operatorname{sym}(\boldsymbol{\xi}_{,\mathbf{g}}) \tag{3.11}$$

It follows from (2.12) and $(3.11)_{1-2}$ that $\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}(\bar{\nu}, \dot{\bar{\nu}}, \bar{\gamma}, \vartheta^M, \vartheta^T, \vartheta^G, \mathbf{g}_{1-5})$. Integrating $(3.11)_3$ one by one, yields (Wang, 1970, 1971; Wang and Liu, 1980)

$$\begin{aligned}\boldsymbol{\xi} &= \mathbf{A}_1\cdot\mathbf{g}_1 + \mathbf{B}_1\cdot\mathbf{g}_2 + \mathbf{C}_1\cdot(\mathbf{g}_1\otimes\mathbf{g}_2) + \mathbf{d}_1 = \mathbf{A}_2\cdot\mathbf{g}_2 + \mathbf{B}_2\cdot\mathbf{g}_3 + \mathbf{C}_2\cdot(\mathbf{g}_2\otimes\mathbf{g}_3) + \mathbf{d}_2 \\ &= \mathbf{A}_3\cdot\mathbf{g}_3 + \mathbf{B}_3\cdot\mathbf{g}_4 + \mathbf{C}_3\cdot(\mathbf{g}_3\otimes\mathbf{g}_4) + \mathbf{d}_3 \\ &= \mathbf{A}_4\cdot\mathbf{g}_4 + \mathbf{B}_4\cdot\mathbf{g}_5 + \mathbf{C}_4\cdot(\mathbf{g}_4\otimes\mathbf{g}_5) + \mathbf{d}_4 = \mathbf{A}_5\cdot\mathbf{g}_1 + \mathbf{B}_5\cdot\mathbf{g}_5 + \mathbf{C}_5\cdot(\mathbf{g}_1\otimes\mathbf{g}_5) + \mathbf{d}_5\end{aligned} \tag{3.12}$$

with $\{\mathbf{A}_{1-5}, \mathbf{B}_{1-5}\}$ and $\mathbf{C}_{1-5}$ being respectively the second- and third-rank skew-symmetric tensors, and $\mathbf{d}_{1-5}$ the isotropic vectors. Since $\boldsymbol{\xi}$ is an isotropic vector, it follows that $\mathbf{0} = \mathbf{A}_{1-5} = \mathbf{B}_{1-5}$, and $\mathbf{0} = \mathbf{C}_{1-5}$, because there are no isotropic second- and third-rank skew-symmetric tensors. With these, equation (3.12) reduces to

$$\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}(\bar{\nu}, \dot{\bar{\nu}}, \bar{\gamma}, \vartheta^M, \vartheta^T, \vartheta^G) = \mathbf{0} \tag{3.13}$$

since there exist no isotropic vectors with only scalar arguments.

Equation (3.13) indicates that for rapid flows, the granular system as a whole and solid grains are moving in a way that combined contributions of two-fold turbulent kinetic energy fluxes are collinear with the entropy flux, a distinct result from that of dense flows (Fang and Wu, 2014a). For in rapid flows, the short-term grain-grain interaction is dominant, causing the grains to evolve in a more dispersive manner from the granular system. In dense flows, the long-term one dominates, resulting in more orientated grain arrangement from the granular body. With $\boldsymbol{\xi} = \mathbf{0}$, equation $(3.8)_2$ reduces to

$$\ell(\vartheta^M\bar{\mathbf{h}} + \vartheta^G\mathbf{H}) = \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\mathbf{g}_1} \qquad \mathbf{H} = \ell\mathbf{R}\mathbf{g}_1 \tag{3.14}$$

With (3.5)-(3.7) and (3.13)-(3.14), equation $(3.4)_1$ has been fully explored.

Residual entropy inequality (3.9) is recast, viz.

$$\begin{aligned}\bar{\pi} = &\Big(\vartheta^M\bar{\nu}\bar{p}\mathbf{I} + \vartheta^M\bar{\mathbf{t}} + \vartheta^T\mathbf{R} + \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\mathbf{g}_1}\otimes\mathbf{g}_1\Big)\cdot\bar{\mathbf{D}} + (\bar{\mathbf{q}}+\mathbf{Q})\cdot\mathbf{g}_3 \\ &- \mathbf{K}\cdot\mathbf{g}_4 - \mathbf{L}\cdot\mathbf{g}_5 + \vartheta^M(\bar{p} - \bar{\beta} - \bar{\gamma}\bar{\nu}\bar{f}\ell)\dot{\bar{\nu}} + \bar{\pi}_{int} \geqslant 0\end{aligned} \tag{3.15}$$

with the internal dissipation $\bar{\pi}_{int}$

$$\bar{\pi}_{int} = \bar{\gamma}\bar{\nu}\varepsilon(\vartheta^M - \vartheta^T) + \bar{\gamma}\bar{\nu}H(\vartheta^M - \vartheta^G) - \bar{\gamma}\bar{\nu}\vartheta^M \psi^T_{,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}} \tag{3.16}$$

and the abbreviations

$$\bar{p} \equiv \bar{\gamma}^2 \psi^T_{,\bar{\gamma}} \qquad \bar{\beta} \equiv \bar{\gamma}\bar{\nu}\psi^T_{,\bar{\nu}} \tag{3.17}$$

known respectively as the *turbulent thermodynamic pressure* and *turbulent configurational pressure*. They are extended versions of their counterparts in laminar flows (Fang, 2009a; Kirchner, 2002; Wang and Hutter, 1999).

4. Equilibrium closure relations

Thermodynamic equilibrium is defined to be a time-independent process with uniform vanishing mean entropy production, viz. (Hutter and Wang, 2003)

$$\bar{\pi}\Big|_{\mathrm{E}} = 0 \tag{4.1}$$

with the subscript E indicating that the indexed quantity is evaluated at thermodynamic equilibrium. In view of (2.12) and (3.15)-(3.16), equation (4.1) motivates the following equilibrium and dynamic state spaces, viz.

$$\begin{aligned} \mathcal{Q}\Big|_{\mathrm{E}} &\equiv (\nu_0, \bar{\nu}, 0, \mathbf{g}_1, \bar{\gamma}, \mathbf{g}_2, \vartheta^M, \mathbf{0}, \vartheta^T, \mathbf{0}, \vartheta^G, \mathbf{0}, \mathbf{0}, \bar{\mathbf{Z}}) \\ \mathcal{Q}^D &\equiv (\dot{\bar{\nu}}, \mathbf{g}_3, \mathbf{g}_4, \mathbf{g}_5, \bar{\mathbf{D}}) \end{aligned} \tag{4.2}$$

with the superscript D denoting the dynamic state space. The dynamic variables ($\dot{\bar{\nu}}$, $\mathbf{g}_3$, $\mathbf{g}_4$, $\mathbf{g}_5$, $\bar{\mathbf{D}}$) should vanish at the equilibrium state, with $\mathcal{Q}^D\Big|_{\mathrm{E}} = 0$. In addition, under sufficient smoothness, $\bar{\pi}$ has to satisfy

$$\bar{\pi}_{,a}\Big|_{\mathrm{E}} = 0 \qquad a \in \mathcal{Q}^D \tag{4.3}$$

and the Hessian matrix of $\bar{\pi}$ with respect to $\mathcal{Q}^D$ at the thermodynamic equilibrium should be positive semi-definite. Since the latter condition constrains the sign of the material parameters in the closure relations, only equations (4.1) and (4.3) will be investigated.

First, applying (4.1) and $(4.2)_1$ to (3.15) and (3.16), yields

$$0 = (\vartheta^M - \vartheta^T)\bar{\gamma}\bar{\nu}\varepsilon\Big|_{\mathrm{E}} + (\vartheta^M - \vartheta^G)\bar{\gamma}\bar{\nu}H\Big|_{\mathrm{E}} - \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}\Big|_{\mathrm{E}} \tag{4.4}$$

indicating that two-fold turbulent dissipations $\bar{\gamma}\bar{\nu}\varepsilon$ and $\bar{\gamma}\bar{\nu}H$ at the equilibrium state result from the internal friction, a justified result. It can be fulfilled with the assumptions

$$0 = \varepsilon\Big|_{\mathrm{E}} \qquad 0 = H\Big|_{\mathrm{E}} \qquad \mathbf{0} = \bar{\mathbf{\Phi}}\Big|_{\mathrm{E}} \tag{4.5}$$

for all productions cease at the thermodynamic equilibrium state.

Secondly, incorporating $(4.2)_2$ and (4.3) into (3.15) and (3.16), results respectively in

$$
\begin{aligned}
0 &= \vartheta^M(\bar{p} - \bar{\beta} - \bar{\gamma}\bar{\nu}\bar{f}\Big|_{\mathrm{E}}\ell) + (\vartheta^M - \vartheta^T)\bar{\gamma}\bar{\nu}\varepsilon_{,\dot{\bar{\nu}}}\Big|_{\mathrm{E}} + (\vartheta^M - \vartheta^G)\bar{\gamma}\bar{\nu}H_{,\dot{\bar{\nu}}}\Big|_{\mathrm{E}} \\
&\quad - \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}_{,\dot{\bar{\nu}}}\Big|_{\mathrm{E}} \\
\mathbf{0} &= (\bar{\mathbf{q}} + \mathbf{Q})\Big|_{\mathrm{E}} + (\vartheta^M - \vartheta^T)\bar{\gamma}\bar{\nu}\varepsilon_{,\mathbf{g}_3}\Big|_{\mathrm{E}} + (\vartheta^M - \vartheta^G)\bar{\gamma}\bar{\nu}H_{,\mathbf{g}_3}\Big|_{\mathrm{E}} - \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}_{,\mathbf{g}_3}\Big|_{\mathrm{E}} \\
\mathbf{0} &= -\mathbf{K}\Big|_{\mathrm{E}} + (\vartheta^M - \vartheta^T)\bar{\gamma}\bar{\nu}\varepsilon_{,\mathbf{g}_4}\Big|_{\mathrm{E}} + (\vartheta^M - \vartheta^G)\bar{\gamma}\bar{\nu}H_{,\mathbf{g}_4}\Big|_{\mathrm{E}} - \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}_{,\mathbf{g}_4}\Big|_{\mathrm{E}} \\
\mathbf{0} &= -\mathbf{L}\Big|_{\mathrm{E}} + (\vartheta^M - \vartheta^T)\bar{\gamma}\bar{\nu}\varepsilon_{,\mathbf{g}_5}\Big|_{\mathrm{E}} + (\vartheta^M - \vartheta^G)\bar{\gamma}\bar{\nu}H_{,\mathbf{g}_5}\Big|_{\mathrm{E}} - \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}_{,\mathbf{g}_5}\Big|_{\mathrm{E}} \\
\mathbf{0} &= \vartheta^M\bar{\mathbf{t}}\Big|_{\mathrm{E}} + \vartheta^T\mathbf{R}\Big|_{\mathrm{E}} + \bar{\nu}\vartheta^M\bar{p}\mathbf{I} + \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\mathbf{g}_1} \otimes \mathbf{g}_1 + (\vartheta^M - \vartheta^T)\bar{\gamma}\bar{\nu}\varepsilon_{,\bar{\mathbf{D}}}\Big|_{\mathrm{E}} \\
&\quad + (\vartheta^M - \vartheta^G)\bar{\gamma}\bar{\nu}H_{,\bar{\mathbf{D}}}\Big|_{\mathrm{E}} - \bar{\gamma}\bar{\nu}\vartheta^M\psi^T_{,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}_{,\bar{\mathbf{D}}}\Big|_{\mathrm{E}}
\end{aligned}
\tag{4.6}
$$

While equation $(4.6)_1$ indicates that the internal friction and two-fold turbulent dissipations affect the evolution of $\bar{\nu}$ via two-fold pressures, justified in view of the microstructural grain-grain interactions, it also yields an equilibrium expression for $\bar{f}$. Equation $(4.6)_2$ delivers that the equilibrium mean and turbulent heat fluxes are related to the internal friction and two-fold turbulent dissipations, according to observations. It reduces to vanishing mean and turbulent heat fluxes for isothermal flows. Equation $(4.6)_3$ delivers an equilibrium expression for $\mathbf{K}$ in terms of the internal friction and two-fold turbulent dissipations. Since turbulent dissipation is (negative) production of $\bar{\gamma}\bar{\nu}k$, $(4.6)_3$ and (4.4) imply an energy cascade from the turbulent kinetic energy flux toward turbulent dissipation through the effect of internal friction in the presence of a non-uniform granular coldness gradient, a phenomenon similar to that of Newtonian fluids in turbulent shear flows (Batchelor, 1993; Tsinober, 2009). A similar situation is also found between $\mathbf{L}$ and $\bar{\gamma}\bar{\nu}H$, as indicated by $(4.6)_4$. Equations $(4.6)_3$ and $(4.6)_4$ also imply that $\mathbf{K}$ and $\mathbf{L}$ are mutually influenced, a result already indicated by (3.13). Lastly, equation $(4.6)_5$ yields an expression that should be satisfied by $\bar{\mathbf{t}}|_E$ and $\mathbf{R}|_E$ in terms of the internal friction, two-fold turbulent dissipations and the mean volume fraction gradient.

In the above, $\bar{\gamma}\bar{\nu}k$ and $\bar{\gamma}\bar{\nu}s$ are expressed respectively as functions of ϑ^T and ϑ^G; they are determined once ψ^T is prescribed. Thus, equations (3.5), $(3.7)_{2-3}$, (3.17), and (4.4)-(4.6) deliver implicitly the equilibrium closure relations for $\bar{\mathbf{h}}$, $\mathbf{H}$, $\bar{f}$, $\bar{\mathbf{t}}$, $\mathbf{R}$, $\mathbf{K}$, $\mathbf{L}$, ε, H and $\bar{\mathbf{\Phi}}$ in the context of the zero-order turbulence closure model.

Remarks

1. The derived equilibrium closure relations apply for both compressible and incompressible grains. For the incompressible grains, $\bar{p}$ is no longer determined by $(3.17)_1$, and should be computed from the momentum equation.
2. Equation (4.5) is made based on observations of Newtonian fluids in turbulent motion, with which equations (2.8) and (2.9) are automatically fulfilled, when $\mathbf{K}$ and $\mathbf{L}$ are assumed to depend explicitly on $\mathbf{g}_4$ and $\mathbf{g}_5$, respectively.
3. In the second part, a hypoplastic model is used for $\bar{\mathbf{\Phi}}$ with $\bar{\mathbf{\Phi}} = \hat{\bar{\mathbf{\Phi}}}(\bar{\nu}, \bar{\mathbf{D}}, \bar{\mathbf{Z}})$, with which equation $(4.5)_3$ is fulfilled, and equations (2.8) and (2.9) reduce to

$$
0 = \nabla \cdot \Big((\vartheta^M - \vartheta^T)\bar{\gamma}\bar{\nu}\varepsilon_{,\mathbf{g}}\Big|_E + (\vartheta^M - \vartheta^G)\bar{\gamma}\bar{\nu}H_{,\mathbf{g}}\Big|_E \Big) \qquad \mathbf{g} = \{\mathbf{g}_4, \mathbf{g}_5\} \tag{4.7}
$$

additionally restrictions that should be fulfilled by $\varepsilon|_{\mathrm{E}}$ and $H|_{\mathrm{E}}$.

4. Equation $(4.6)_5$, by using Truesdell's equi-presence principle, is decomposed into

$$\begin{aligned} \vartheta^M \bar{\mathbf{t}}\Big|_{\mathrm{E}} &= -\bar{\nu}\vartheta^M \bar{p}\mathbf{I} - \bar{\gamma}\bar{\nu}\vartheta^M \psi^T_{,\mathbf{g}_1} \otimes \mathbf{g}_1 + \bar{\gamma}\bar{\nu}\vartheta^M \psi^T_{,\bar{\mathbf{Z}}} \cdot \bar{\mathbf{\Phi}}_{,\bar{\mathbf{D}}}\Big|_{\mathrm{E}} \\ \vartheta^T \mathbf{R}\Big|_{\mathrm{E}} &= -(\vartheta^M - \vartheta^T)\bar{\gamma}\bar{\nu}\varepsilon_{,\bar{\mathbf{D}}}\Big|_{\mathrm{E}} - (\vartheta^M - \vartheta^G)\bar{\gamma}\bar{\nu}H_{,\bar{\mathbf{D}}}\Big|_{\mathrm{E}} \end{aligned} \tag{4.8}$$

in which $\bar{\mathbf{t}}\Big|_{\mathrm{E}}$ is generated through the mean fields, with $\mathbf{R}|_{\mathrm{E}}$ mainly induced via the quantities related to turbulent fluctuation (e.g. two-fold turbulent dissipations), a procedure widely used for Newtonian fluids in turbulent flows. For laminar flows, equation $(4.8)_2$ yields a vanishing $\mathbf{R}|_{\mathrm{E}}$, while equation $(4.8)_1$ delivers that a dry granular heap at the equilibrium state can be accomplished either by the internal friction or a non-uniform gradient of $\bar{\nu}$, coinciding with the previous works by Fang (2009a), Kirchner (2002), Wang and Hutter (1999).

5. Concluding remarks

The Reynolds-filter process is used to decompose the variables into the mean and fluctuating parts to obtain the balance equations of the mean fields. Two-fold granular coldnesses are introduced as primitive fields to index the variations in the turbulent kinetic and turbulent configurational kinetic energies, with their dissipations considered closure relations. The Müller-Liu entropy principle is investigated to derive the equilibrium closure relations.

Equations (3.13) and $(4.6)_{3,4}$ demonstrate that the flux of the turbulent kinetic energy and that of the turbulent configurational kinetic energy are mutually influenced. This implies that the turbulent fluctuation of the granular system as a whole tends to drive the grains to arrange and fluctuate in a way that their combined contributions provide the only deviation between the heat and entropy fluxes. This is justified, since laboratory experiments and field observations suggest that there exists a turbulent boundary layer immediately above the base, in which the grains collide one with another vigorously, resulting in a dominant short-term grain-grain interaction. On the other hand, this conclusion does not hold for dense flows with weak turbulent intensity, for only the collinearity between the fluxes of the turbulent kinetic energy and evolution of the mean volume fraction can be deduced.

The implementation of the closure model, and numerical simulation of an isothermal, isochoric, gravity-driven stationary dry granular avalanche with incompressible grains down an incline, compared with the laminar flow solutions are reported in the second part.

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