JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
55, 3, pp. 1015-1027, Warsaw 2017
DOI: 10.15632/jtam-pl.55.3.1015

NONCONSERVATIVE STABILITY OF VISCOELASTIC PLATES SUBJECT TO TRIANGULARLY DISTRIBUTED FOLLOWER LOADS

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Divergence and flutter instabilities of viscoelastic rectangular plates under triangularly distributed tangential follower loads are studied. Two sets of boundary conditions are considered, namely, simply supported plates and plates with a combination of clamped and simply supported edges. The constitutive relations for the viscoelastic plates are of Kelvin-Voigt type with the effect of viscoelasticity on stability studied numerically. The method of solution is differential quadrature which is employed to discretize the equation of motion and the boundary conditions leading to a generalized eigenvalue problem. After verifying the method of solution, numerical results are given for the real and imaginary parts of the eigenfrequencies to investigate flutter and divergence characteristics and dynamic stability of the plates with respect to various problem parameters.

Keywords: viscoelastic plates, dynamic stability, triangularly distributed follower load

1. Introduction

Dynamic stability of elastic structures subject to nonconservative loads is of practical importance in such fields as aerospace, mechanical and civil engineering. As a result, the subject has been studied extensively to quantify the behaviour of beams, plates and shells under follower forces. These forces can be concentrated, uniformly distributed or triangularly distributed depending on the specific application. They act in the tangential direction and are not derivable from a potential due to their nonconservative nature as presented in works by Kumar and Srivasta (1986), Przybylski (1999), Gajewski (2000), Krillov (2013).

Early work on the nonconservative instability under uniformly distributed follower loads mostly involved one dimensional elastic structures, namely, columns (Sugiyama and Kawagoe, 1975; Leipholz, 1975; Chen and Ku, 1991). Stability of columns under triangularly distributed loads was studied by Leipholz and Bhalla (1977), Sugiyama and Mladenov (1983) and Ryu *et al.* (2000). More recent studies on nonconservative loading include columns subject to uniformly distributed follower loads by Kim (2010), Kim *et al.* (2008) and Kazemi-Lari *et al.* (2013) and to triangularly distributed follower loads by Kim (2011). The follower force can also be realized by means of properly shaped loading heads which can affect the displacements of the loaded end as studied by Tomski and Szmidla (2004) and Tomski and Uzny (2013b). The installation of Tomski's head can lead to a loading force which is tangent to the end of the column and can be conservative (Tomski and Uzny, 2008, 2013a). The force in this case is directed to a constant point which becomes a pole for the loading. Introduction of Tomski's head can lead to new characteristic curves such as pseudo flutter and allows one to control the dynamic behavior of the system. Studies on nonconservative stability of two-dimensional structures mostly involved rectangular plates under follower loads (Culkowski and Reismann, 1977; Farshad, 1978; Adali,

1982) and under uniformly distributed tangential loads (Leipholz, 1978; Leipholz and Pfendt, 1982, 1983; Wang and Ji, 1992).

Recent research on the stability of elastic plates under nonconservative loads includes works by Zuo and Shreyer (1996), Kim and Park (1998), Kim and Kim (2000), and Jayaraman and Struthers (2005). Dynamic stability of functionally graded plates under uniformly distributed axial loads was studied by Ruan *et al.* (2012) and shells by Torki *et al.* (2014a,b). These studies neglected the effect of viscoelasticity on the stability of the columns and plates.

Dynamic stability of one-dimensional viscoelastic structures was the subject of the works by Marzani and Potapov (1999), Langthjem and Sugiyama (2000), Darabseh and Genin (2004), Zhuo and Fen (2005), Ilyasov and Ilyasova (2006) and Elfelsoufi and Azrar (2006). Recently, the dynamic stability of viscoelastic plates has been studied for a number of cases (Ilyasov and Aköz, 2000; Wang *et al.*, 2007, 2009, 2013; Zhou and Wang, 2014; Robinson and Adali, 2016). Vibrations of a simply supported plate with nonlinear strain-displacement relations and subject to a uniformly distributed tangential force were studied by Robinson (2013). Dynamic stability of viscoelastic shells was studied by Ilyasov (2010).

Although the dynamic stability under triangularly distributed tangential forces have been studied in the case of columns (see Leipholz and Bhalla, 1977; Sugiyama and Mladenov, 1983; Ryu *et al.*, 2000; Kim, 2011), dynamic stability of plates and, in particular, viscoelastic plates under this type of loading does not seem to be studied so far.

The present work extends the results of Robinson and Adali (2016) who studied nonconservative stability of viscoelastic plates with free edges and under uniformly distributed follower loads, to the case of plates with simply supported and simply supported-clamped plates and subject to triangularly distributed follower loads. Comparisons are given for the uniformly and triangularly distributed follower loads. The stability problem is solved for the simply supported plates and for plates with a combination of simple and clamped supports by the differential quadrature method. Divergence and flutter loads are determined and the effect of viscoelasticity and the boundary conditions on dynamic stability is investigated. The method of solution is verified against the known results in the literature.

2. Viscoelastic plate subject to triangularly distributed load

We consider a rectangular plate of uniform thickness h having dimensions $a \times b$ in the x and y directions, respectively (see Fig. 1). It is subject to a non-uniform tangential follower force $q_t = q_0 a(1 - x/a)$.

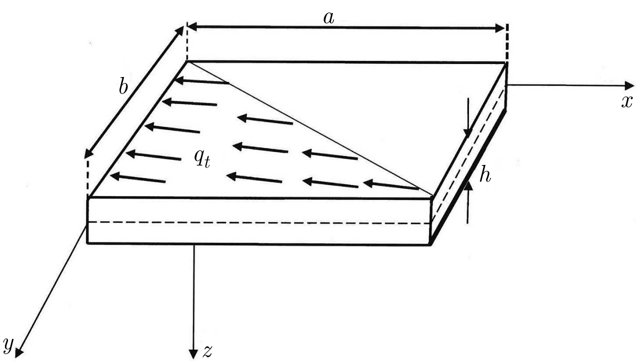

Fig. 1. Viscoelastic plate subjected to a triangular follower load

The material of the plate is viscoelastic which is expressed by the Kelvin-Voigt constitutive relations given by

$$\mathbf{s}_{ij} = 2G\mathbf{e}_{ij} + 2\eta\dot{\mathbf{e}}_{ij} \qquad \boldsymbol{\sigma}_{ii} = 3K\boldsymbol{\varepsilon}_{ii} \tag{2.1}$$

where $\mathbf{s}_{ij}$ and $\mathbf{e}_{ij}$ are deviatoric tensors of stress and strain, respectively, and $\boldsymbol{\sigma}_{ii}$ and $\boldsymbol{\varepsilon}_{ii}$ are spherical tensors of stress and strain with η denoting the viscoelastic coefficient. The bulk modulus K and shear modulus G can be expressed in terms of Young's modulus E and Poisson's ratio ν as $K = E/3(1-2\nu)$ and $G = E/(1+2\nu)$. The equation of vibration of the viscoelastic plate subject to a triangular follower load is first obtained in the Laplace domain (see Wang *et al.*, 2007; Zhou and Wang, 2014). By the inverse Laplace transformation, the governing equation can be expressed in the time domain as

$$\frac{h^3}{12}\Big(A_3 + A_4\frac{\partial}{\partial t} + A_5\frac{\partial^2}{\partial t^2}\Big)\nabla^4 w + \Big(A_1 + A_2\frac{\partial}{\partial t}\Big)\Big(\frac{q_0(a-x)^2}{2}\frac{\partial^2 w}{\partial x^2} + \rho h\frac{\partial^2 w}{\partial t^2}\Big) = 0 \tag{2.2}$$

where ρ is density of the plate and

$$\begin{array}{lll} A_1 = 3K + 4G & A_1 = 3K + 4G & A_2 = 4\eta \\ A_3 = 4G(3K+G) & A_4 = \eta(8G + 12K) & A_5 = 4\eta^2 \end{array} \tag{2.3}$$

and

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \tag{2.4}$$

After introducing dimensionless coefficients

$$X = \frac{x}{a} \qquad Y = \frac{y}{b} \qquad \overline{w} = \frac{w}{h} \qquad \lambda = \frac{a}{b} \tag{2.5}$$

and

$$q = \frac{6q_0 a^4(1-\nu^2)}{Eh^3} \qquad \tau = \frac{th}{a^2}\sqrt{\frac{E}{12\rho(1-\nu^2)}} \qquad H = \frac{\eta h}{a^2}\sqrt{\frac{1}{12\rho(1-\nu^2)E}} \tag{2.6}$$

the non-dimensional equation of motion is obtained as

$$\Big(1 + g_1\frac{\partial}{\partial \tau} + g_2\frac{\partial^2}{\partial \tau^2}\Big)\nabla^4\overline{w} + \Big(1 + g_3\frac{\partial}{\partial \tau}\Big)\Big(q(1-X)^2\frac{\partial^2 \overline{w}}{\partial X^2} + \frac{\partial^2 \overline{w}}{\partial \tau^2}\Big) = 0 \tag{2.7}$$

where

$$\begin{array}{ll} g_2 = \dfrac{4(1-2\nu)(1+\nu)^2}{3}H^2 & g_3 = \dfrac{4(1-2\nu)(1+\nu)}{3(1-\nu)}H \\ \nabla^4\overline{w} = \dfrac{\partial^4 \overline{w}}{\partial X^4} + 2\lambda^2\dfrac{\partial^4 \overline{w}}{\partial X^2 \partial Y^2} + \lambda^4\dfrac{\partial^4 \overline{w}}{\partial Y^4} & \end{array} \tag{2.8}$$

In equations (2.8), H is the dimensionless delay time of the material and τ is the dimensionless time defined in Eq. (2.6). Let

$$\overline{w}(X,Y,\tau) = W(X,Y)\mathrm{e}^{\mathrm{j}\omega\tau} \tag{2.9}$$

where $\mathrm{j} = \sqrt{-1}$ and ω the dimensionless vibration frequency. Substitution of equation (2.9) into equation (2.7) yields the differential equation

$$(1 + g_1\mathrm{j}\omega + g_2\mathrm{j}^2\omega^2)\nabla^4 W + (1 + g_3\mathrm{j}\omega)\Big(q(1-X)^2\frac{\partial^2 W}{\partial X^2} + \mathrm{j}^2\omega^2\Big) = 0 \tag{2.10}$$

in terms of the space variables X and Y. The boundary conditions considered in the present work are the simply supported plates (SSSS) and plates with two opposite edges clamped and two others simply supported (CSCS).

SSSS boundary conditions are given by

$$\overline{w}(X,Y,\tau)=\begin{cases}\left.\dfrac{\partial^2\overline{w}}{\partial X^2}\right|_{X=0,1}=0 & \text{for} \quad 0\leqslant Y\leqslant 1\\ \left.\dfrac{\partial^2\overline{w}}{\partial Y^2}\right|_{Y=0,1}=0 & \text{for} \quad 0\leqslant X\leqslant 1\end{cases} \tag{2.11}$$

CSCS boundary conditions are given by

$$\overline{w}(X,Y,\tau)=\begin{cases}\left.\dfrac{\partial\overline{w}}{\partial X}\right|_{X=0,1}=0 & \text{for} \quad 0\leqslant Y\leqslant 1\\ \left.\dfrac{\partial^2\overline{w}}{\partial Y^2}\right|_{Y=0,1}=0 & \text{for} \quad 0\leqslant X\leqslant 1\end{cases} \tag{2.12}$$

3. Differential quadrature (DQ) method

The DQ method involves approximating the partial derivatives of the function $W(X,Y)$ at a sample point (X_i,Y_j) by the weighted sum of the function W_{ij} (see Bert and Malik, 1996; Krowiak, 2008). Let the number of sample points denoted by N in the X direction and M in the Y direction. The r-th order partial derivative with respect to X, s-th order partial derivative with respect to Y and the $(r+s)$-th order mixed partial derivative of $W(X,Y)$ with respect to both X and Y are discretely expressed at the point (X_i,Y_j) as

$$\begin{aligned}&\frac{\partial^r W(X_i,Y_j)}{\partial X^r}=\sum_{k=1}^{N}A_{ik}^{(r)}W_{kj} \qquad \frac{\partial^s W(X_i,Y_j)}{\partial Y^s}=\sum_{k=1}^{M}B_{jl}^{(s)}W_{il}\\ &\frac{\partial^{r+s} W(X_i,Y_j)}{\partial X^r\partial Y^s}=\sum_{k=1}^{N}A_{ik}^{(r)}\sum_{l=1}^{M}B_{jl}^{(s)}W_{kl}\end{aligned} \tag{3.1}$$

where $i=1,2,\ldots,N$, $k=1,2,\ldots,N-1$, $j=1,2,\ldots,M$ and $l=1,2,\ldots,M-1$.

For $r=s=1$, the coefficients $A_{ik}^{(r)}$ and $B_{jl}^{(s)}$ are defined as

$$\begin{aligned}A_{ik}^{(1)}&=\begin{cases}\prod\limits_{\mu=1,\mu\neq i}^{N}\dfrac{X_i-X_\mu}{(X_i-X_k)\prod\limits_{\mu-1,\mu\neq k}^{N}(X_k-X_\mu)} & \text{for} \quad i,k=1,2,\ldots,N \quad (i\neq k)\\ \sum\limits_{\mu-1,\mu\neq k}^{N}\dfrac{1}{X_i-X_\mu} & \text{for} \quad i=1,2,\ldots,N \quad (i=k)\end{cases}\\ B_{jl}^{(1)}&=\begin{cases}\prod\limits_{\mu=1,\mu\neq j}^{M}\dfrac{Y_j-Y_\mu}{(Y_j-Y_l)\prod\limits_{\mu-1,\mu\neq l}^{M}(Y_l-Y_\mu)} & \text{for} \quad j,l=1,2,\ldots,M \quad (j\neq l)\\ \sum\limits_{\mu=1,\mu\neq j}^{M}\dfrac{1}{Y_j-Y_\mu} & \text{for} \quad j=1,2,\ldots,M \quad (j=l)\end{cases}\end{aligned} \tag{3.2}$$

For $r = 2,3,\ldots,N-1$ and $s = 2,3,\ldots,M-1$

$$A_{ik}^{(r)} = \begin{cases} r\Big(A_{ii}^{(r-1)}A_{ik}^{(1)} - \dfrac{A_{ik}^{(r-1)}}{X_i - X_k}\Big) & \text{for} \quad i,k = 1,2,\ldots,N \quad (i \neq k) \\ -\sum\limits_{\mu=1,\mu\neq i}^{N} A_{i\mu}^{(r)} & \text{for} \quad i = 1,2,\ldots,N \quad (i = k) \end{cases}$$
$$B_{jl}^{(s)} = \begin{cases} s\Big(B_{jj}^{(s-1)}B_{jl}^{(1)} - \dfrac{B_{jl}^{(s-1)}}{Y_j - Y_l}\Big) & \text{for} \quad j,l = 1,2,\ldots,M \quad (j \neq l) \\ -\sum\limits_{\mu=1,\mu\neq j}^{M} B_{j\mu}^{(s)} & \text{for} \quad j = 1,2,\ldots,M \quad (j = l) \end{cases} \tag{3.3}$$

The distribution of the grid points are taken as non-uniform, and for the simply supported plate, the grid points are specified as

$$\begin{aligned} &X_1 = 0 \qquad X_N = 1 \qquad X_i = \frac{1}{2}\Big[1 - \cos\Big(\frac{2i-3}{2N-4}\pi\Big)\Big] \qquad \text{for} \quad i = 2,3,\ldots,N-1 \\ &Y_1 = 0 \qquad Y_M = 1 \qquad Y_j = \frac{1}{2}\Big[1 - \cos\Big(\frac{2j-3}{2N-4}\pi\Big)\Big] \qquad \text{for} \quad j = 2,3,\ldots,M-1 \end{aligned} \tag{3.4}$$

For the plate with two opposite edges simply supported and other two edges clamped, the δ method combined with the weighted coefficient method is adopted. Thus, the grid points for the CSCS plate are given by

$$\begin{aligned} &X_1 = 0 \qquad X_2 = \delta \qquad X_{N-1} = 1 - \delta \qquad X_N = 1 \\ &X_i = \frac{1}{2}\Big[1 - \cos\Big(\frac{i-2}{N-3}\pi\Big)\Big] \qquad \text{for} \quad i = 3,4,\ldots,N-2 \\ &Y_1 = 0 \qquad Y_M = 1 \qquad Y_j = \frac{1}{2}\Big[1 - \cos\Big(\frac{2j-3}{2N-4}\pi\Big)\Big] \qquad \text{for} \quad j = 2,3,\ldots,M-1 \end{aligned} \tag{3.5}$$

where $\delta \ll 1$. Using equation (3.1), the discretized form of differential equation (2.10) can be expressed as

$$\begin{aligned} &c_1\mathrm{j}^3W_{ij}\omega^3 + (c_2S_{ij} + W_{ij})\mathrm{j}^2\omega^2 + \Big(c_3S_{ij} \\ &\qquad + c_1q(1-X)^2\sum_{k=1}^{N} A_{ik}^{(2)}W_{kj}\Big)\mathrm{j}\omega + q(1-X)^2\sum_{k=1}^{N} A_{ik}^{(2)}W_{kj} = 0 \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} &S_{ij} = \sum_{k=1}^{N} A_{ik}^{(4)}W_{kj} + 2\lambda^2\sum_{l=1}^{M} B_{jl}^{(2)}\sum_{k=1}^{N} A_{ik}^{(2)}W_{kl} + \lambda^4\sum_{l=1}^{M} B_{jl}^{(4)}W_{il} \\ &c_1 = \frac{4(1-2\nu)(1+\nu)}{3(1-\nu)}H \qquad c_2 = \frac{4(1-2\nu)(1+\nu)^2}{3}H^2 \qquad c_3 = \frac{4(1-2\nu)(1+\nu)}{3}H \end{aligned}$$

The discretized form of boundary conditions (2.11) are given by

$$\begin{aligned} &W_{1j} = W_{Nj} = W_{i1} = W_{iM} = 0 \qquad \text{for} \quad i = 1,2,\ldots,N \quad \wedge \quad j = 1,2,\ldots,M \\ &\sum_{k=1}^{N} A_{ik}^{(2)}W_{kj} = 0 \qquad \text{for} \quad i = 1,N \quad \wedge \quad j = 1,2,\ldots,M \\ &\sum_{l=1}^{M} B_{jl}^{(2)}W_{il} = 0 \qquad \text{for} \quad i = 1,2,\ldots,N \quad \wedge \quad j = 1,M \end{aligned} \tag{3.7}$$

The corresponding equations for boundary conditions (2.12) are

$$
\begin{aligned}
&W_{1j} = W_{Nj} = W_{i1} = W_{iM} = 0 \qquad \text{for} \quad i = 1, 2, \dots, N \quad \wedge \quad j = 1, 2, \dots, M \\
&\sum_{k=1}^{N} A_{ik}^{(1)} W_{kj} = 0 \qquad \text{for} \quad i = 1, 2, \dots, N-1 \quad \wedge \quad j = 2, 3, \dots, M-2 \\
&\sum_{l=1}^{M} B_{jl}^{(2)} W_{il} = 0 \qquad \text{for} \quad i = 1, 2, \dots, N \quad \wedge \quad j = 1, M
\end{aligned}
\tag{3.8}
$$

4. Numerical results and discussion

The results for the viscoelastic plate subject to a triangularly distributed tangential force are given in comparison to the results for a viscoelastic plate subject to a uniformly distributed tangential force which was studied in Wang *et al.* (2007) and Zhou and Wang (2014). The results for the SSSS and CSCS boundary conditions are given in Table 1 for $H = 10^{-5}$ (nondimensional viscoelasticity coefficient). Table 1 shows that the flutter load, denoted by q_f, is higher in the case of the load having triangular distribution as expected. In Table 1, q_{d1} and q_{d2} denote the divergence loads of the 1st and 2nd modes, respectively.

Table 1. Comparison of flutter loads q of viscoelastic plates with $H = 10^{-5}$ for various aspect ratios

Aspect ratio λ	Boundary conditions	Uniformly distributed load, Wang *et al.* (2007)	Triangularly distributed load
1.0	SSSS	$q_{d1} = 67.5$ $q_{d2} = 132.1$	$q_{d1} = 95.1$ $q_{d2} = 225.1$
	CSCS	$q_{d1} = 143.5$ $q_f = 168.0$	$q_f = 226.0$
1.5	SSSS	$q_{d1} = 136.8$ $q_{d2} = 224.7$	$q_{d1} = 174.0$ $q_{d2} = 329.0$
	CSCS	$q_f = 202.8$	$q_f = 270.0$
2.0	SSSS	$q_{d1} = 224.8$ $q_{d2} = 340.5$	$q_{d1} = 273.04$ $q_{d2} = 453.2$
	CSCS	$q_f = 251.5$	$q_f = 333.0$

Figures 2-4 show the real and the imaginary parts of the first three frequencies plotted against the load q for uniformly and triangularly distributed tangential loads for the SSSS plates with $H = 10^{-5}$ and $\lambda = 1$, $\lambda = 1.5$ and $\lambda = 2$, respectively. The corresponding results for the imaginary part of the frequencies for $H = 10^{-3}$ are given in Figs. 5 and 6. It is noted that the results given in Figs. 2-6 for the uniformly distributed tangential load are the same as the ones given in Wang *et al.* (2007). As such, they provide the verification of the method of solution outlined in Section 3.

Comparisons of the loads with uniform and triangular distributions indicate that the results are qualitatively similar, but the magnitudes of the follower load causing divergence or flutter instability differ considerably. Comparisons between Figs. 2a, 3a, 4a ($H = 10^{-5}$) and Figs. 5a, 5b and 6 ($H = 10^{-3}$) indicate that the imaginary parts of the frequencies remain positive for $H = 10^{-3}$ up to the flutter load. The corresponding results for the CSCS plates with $H = 10^{-5}$ are given in Figs. 7-9 with $\lambda = 1$, $\lambda = 1.5$ and $\lambda = 2$, respectively. The results for the uniformly distributed tangential loads are also shown in the figures which verify the results of Wang *et al.*

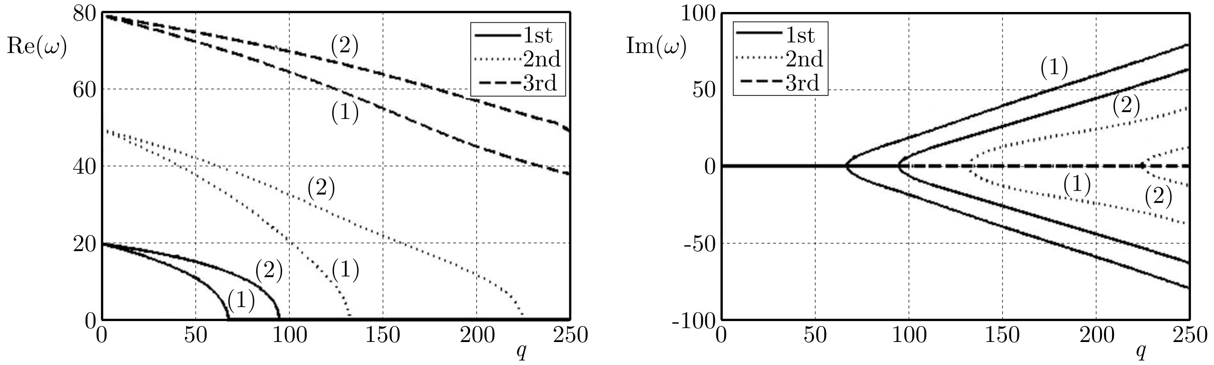

Fig. 2. First three frequencies of SSSS plate vs. follower force for $\lambda = 1$, $H = 10^{-5}$; (1) uniformly distributed load, (2) triangularly distributed load

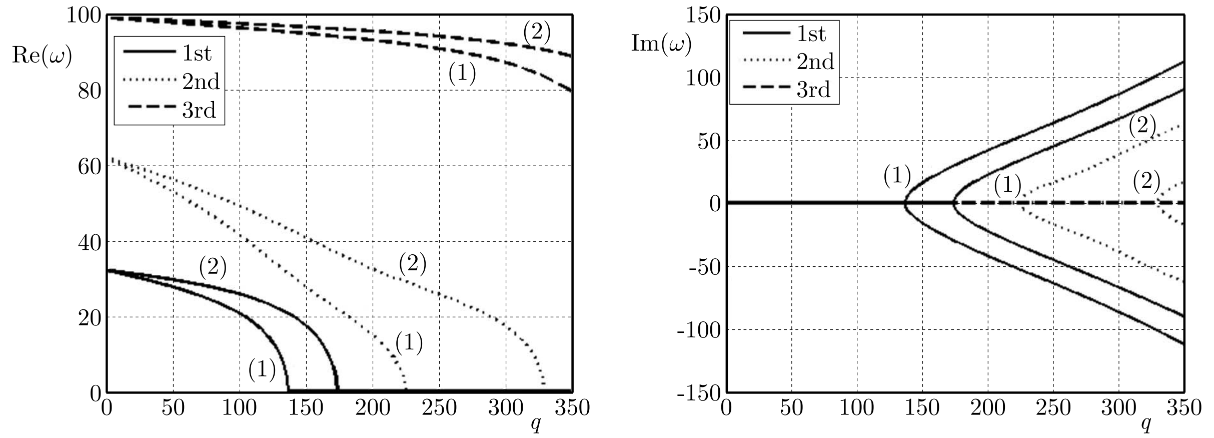

Fig. 3. First three frequencies of SSSS plate vs. follower force for $\lambda = 1.5$, $H = 10^{-5}$; (1) uniformly distributed load, (2) triangularly distributed load

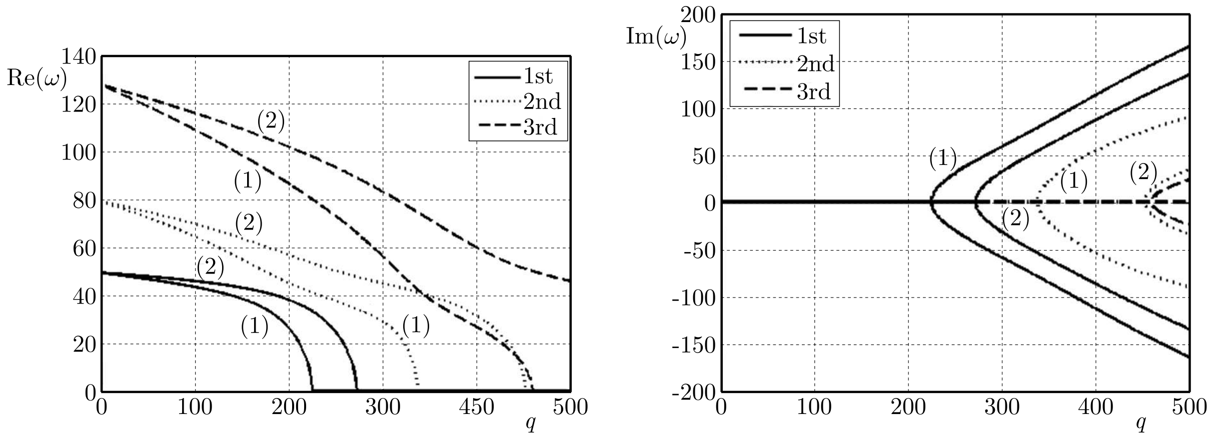

Fig. 4. First three frequencies of SSSS plate vs. follower force for $\lambda = 2$, $H = 10^{-5}$; (1) uniformly distributed load, (2) triangularly distributed load

(2007). In this case, it is observed that the real parts of the vibration modes behave differently as compared to the SSSS plates shown in Figs. 2-4. For the case $\lambda = 1$ (Fig. 7a), the real parts of the first and the third modes join to form a single mode. For $\lambda = 1.5$ and $\lambda = 2$, the first and the second modes join as shown in Figs. 8a and 9a, respectively. Thus, in the case of CSCS boundary conditions, there exists a threshold value q above which the first mode can join the second or third mode to form a single mode, and this value depends on the aspect ratio. Moreover, it is

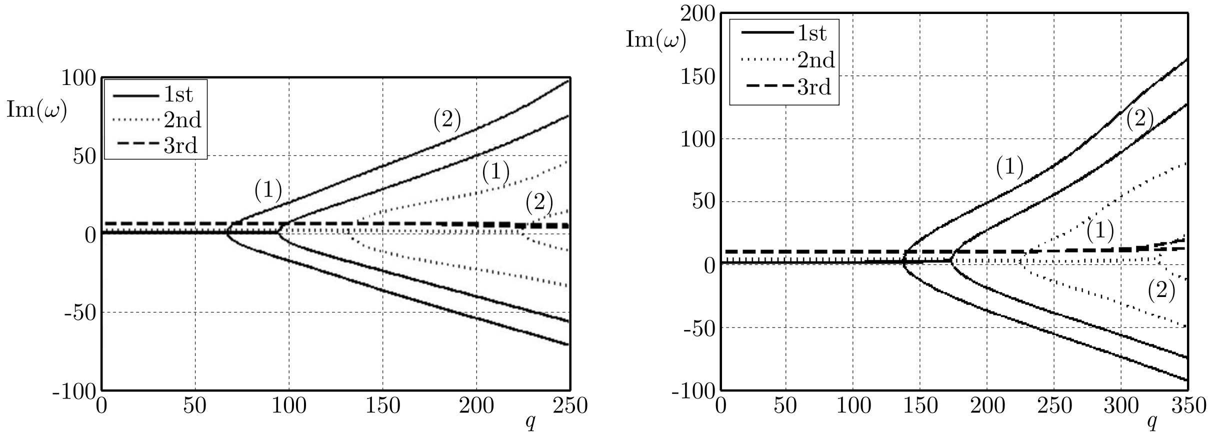

Fig. 5. Imaginary parts of frequencies of SSSS plate vs. follower force for (a) $\lambda = 1$ and (b) $\lambda = 1.5$, $H = 10^{-3}$; (1) uniformly distributed load, (2) triangularly distributed load

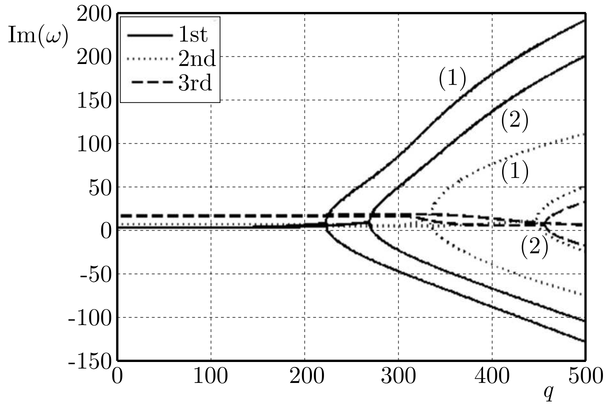

Fig. 6. Imaginary part of frequency of SSSS plate vs. follower force for $\lambda = 2$, $H = 10^{-3}$; (1) uniformly distributed load, (2) triangularly distributed load

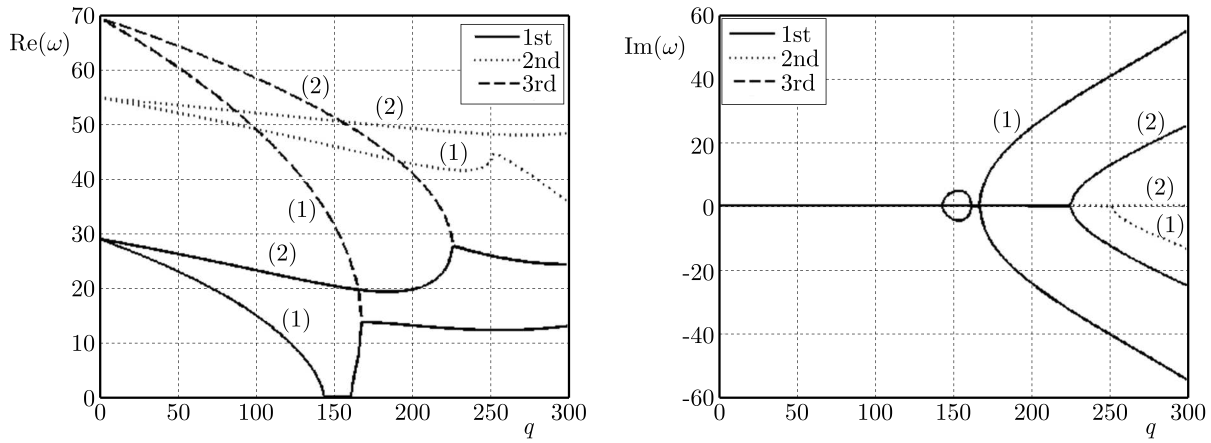

Fig. 7. First three frequencies of CSCS plate vs. follower force for $\lambda = 1$, $H = 10^{-5}$; (1) uniformly distributed load, (2) triangularly distributed load

observed that for the aspect ratios of $\lambda = 1.5$ and $\lambda = 2$, the plate does not show divergence instability and loses stability by flutter.

For the CSCS boundary conditions with $H = 10^{-3}$, the results are given in Figs. 10-12. For this value of $H = 10^{-3}$, the real parts of the frequencies do not form a single mode and the imaginary parts remain positive until the threshold values are exceeded and the flutter

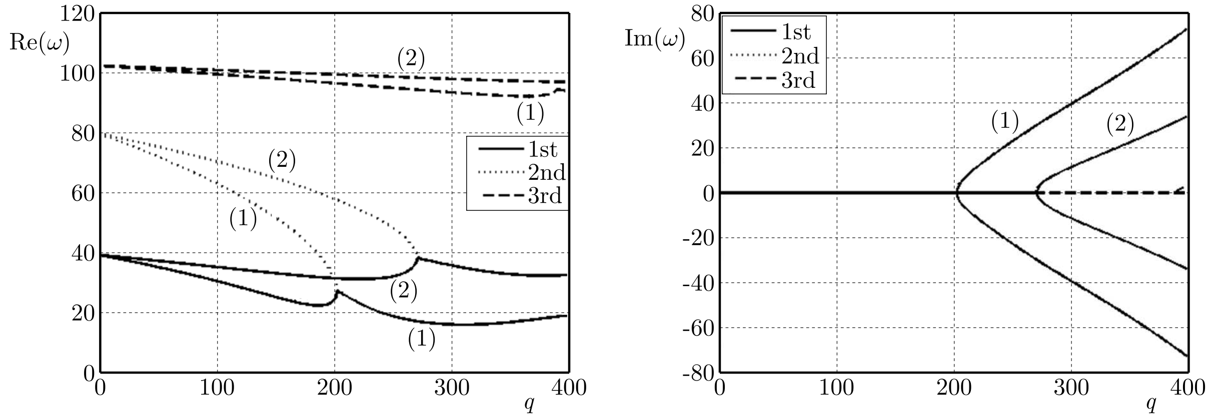

Fig. 8. First three frequencies of CSCS plate vs. follower force for $\lambda = 1.5$, $H = 10^{-5}$; (1) uniformly distributed load, (2) triangularly distributed load

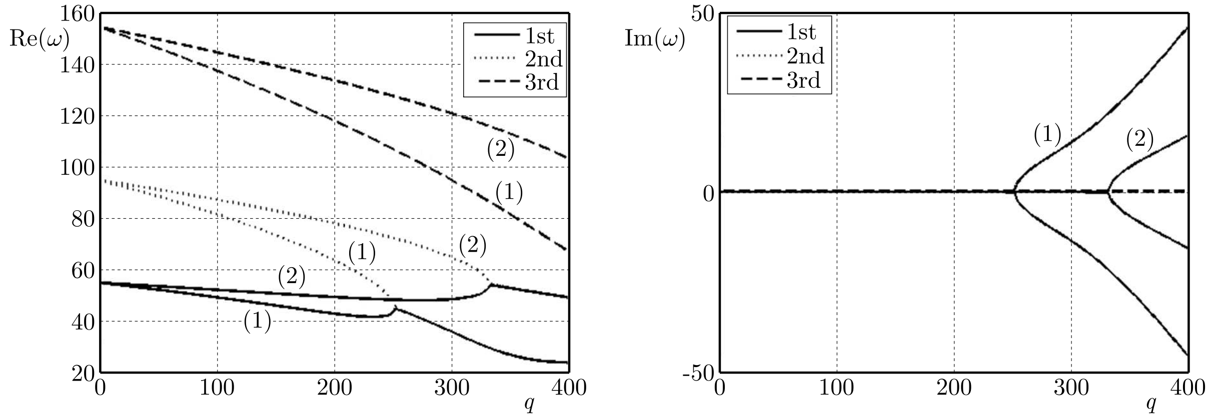

Fig. 9. First three frequencies of CSCS plate vs. follower force for $\lambda = 2$, $H = 10^{-5}$; (1) uniformly distributed load, (2) triangularly distributed load

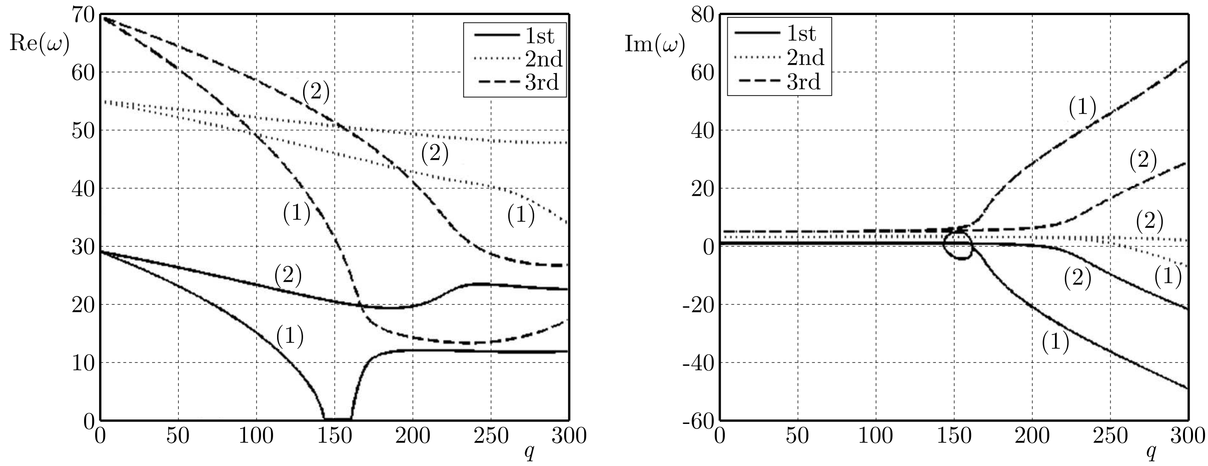

Fig. 10. First three frequencies of CSCS plate vs. follower force for $\lambda = 1$, $H = 10^{-3}$; (1) uniformly distributed load, (2) triangularly distributed load

instability occurs as shown in Figs. 10b, 11b and 12b. The imaginary parts of the frequencies exhibit negative values for $q \geqslant q_f$ leading to an exponential growth of the deflection.

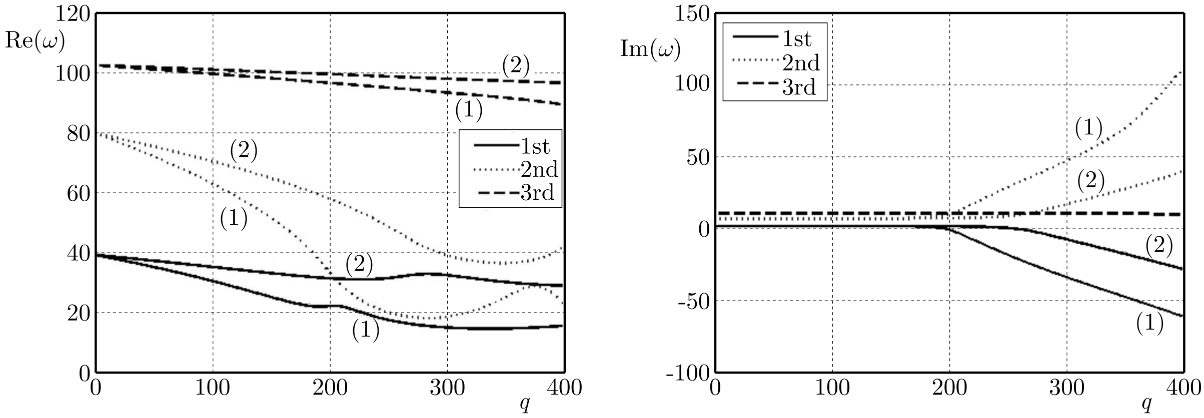

Fig. 11. First three frequencies of CSCS plate vs. follower force for $\lambda = 1.5$, $H = 10^{-3}$; (1) uniformly distributed load, (2) triangularly distributed load

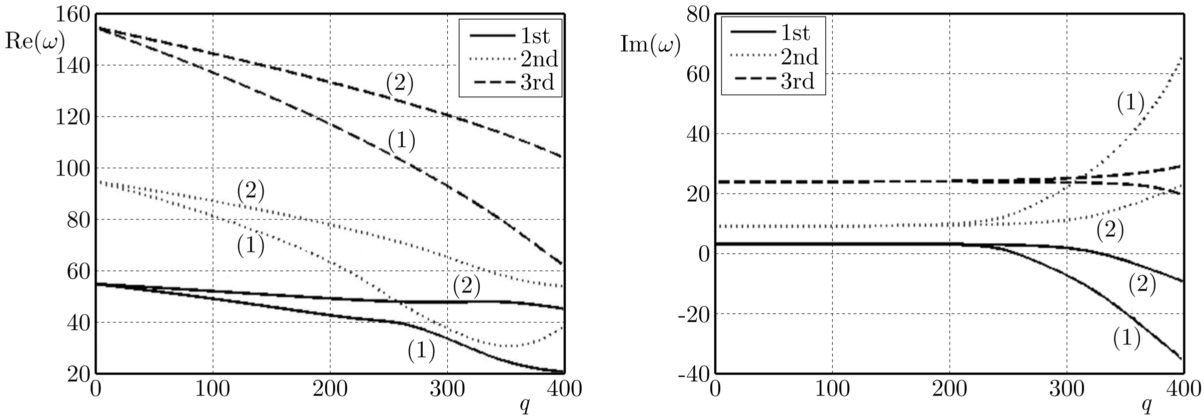

Fig. 12. First three frequencies of CSCS plate vs. follower force for $\lambda = 2$, $H = 10^{-3}$; (1) uniformly distributed load, (2) triangularly distributed load

5. Conclusions

The differential quadrature method is employed to study the dynamic stability of rectangular viscoelastic plates subject to triangularly distributed tangential follower loads. The Kelvin-Voigt viscoelastic model is taken as the constitutive equation of the plate. Two boundary conditions are investigated, namely, simple supports and a combination of simple and fixed supports. The solution is verified against the previous results obtained for SSSS and CSCS viscoelastic plates subject to uniformly distributed tangential loads by Wang *et al.* (2007).

Numerical results are given to study the effects of the aspect ratio and degree of viscoelasticity on the real and imaginary parts of the frequencies. The effect of uniformly and triangularly distributed follower loads on dynamic stability is compared numerically. It is observed that in the case of CSCS plates, the flutter instability occurs before the divergence instability for higher aspect ratios. In the case of SSSS plates, the degree of viscoelasticity does not affect the divergence load, but this effect is more pronounced for CSCS plates. At higher levels of viscoelasticity (higher values of H), the imaginary parts of the complex frequencies become positive rather than zero for low values of the follower load. The results obtained for the present case can be extended to different follower load cases and, in particular, to the cases where the direction of the load is controlled by a head (Tomski and Uzny, 2013b).

Acknowledgment

The research reported in this paper was supported by research grants from the University of KwaZulu-Natal (UKZN) and from National Research Foundation (NRF) of South Africa. The authors gratefully acknowledge the support provided by UKZN and NRF.

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Manuscript received August 3, 2016; accepted for print April 7, 2017