JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
55, 1, pp. 129-139, Warsaw 2017
DOI: 10.15632/jtam-pl.55.1.129

AN ENERGY-BASED METHOD IN PHENOMENOLOGICAL DESCRIPTION OF MECHANICAL PROPERTIES OF NONLINEAR MATERIALS UNDER PLANE STRESS

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A method based on energy is a very useful tool for description of mechanical properties of materials. In the current paper, on the base of geometrical interpretation of a deformation process, the strain energy density function for isotropic nonlinear materials has been constructed. On account of hydrostatic interpretation of the volumetric deformation, the elastic part of energy has been extracted. The initiation of the damage process due to plastic flow of the material under plane stress has been determined and the stability conditions have been formulated by using in the stability analysis the strain energy density function in addition to Sylvester's theorem and assumption of zero volume change during pure plastic deformations. This concept is an original part of the work and continuation of the investigations previously carried out by Wegner and Kurpisz. The theoretical investigations have been illustrated on the example of aluminium.

Keywords: energy-based method, nonlinear material, phenomenological description, strain energy density function, Sylvester's theorem

1. Introduction

The phenomenological description of mechanical properties of nonlinear materials is interesting due to possibility of its application to real engineering structures. The knowledge about mechanisms of deformation under the influence of external loads and the relations between them can be very important in the design process. There is a lot of publications devoted to the modelling of mechanical properties based both on the ground of microstructure relations in the material (multi-scale modelling methods), see for example Silva *et al.* (2007), Terada *et al.* (2008), Speirs *et al.* (2008), or on a phenomenological concept, see for example Wegner (2000, 2005). Because we consider mechanical properties on the base of a real experiment in which the object (material sample) and measured properties are usually in macro-scale, so using the phenomenological method as the way of modelling of the process, which is directly connected with the experiment, is recommended. From this point of view, very interesting are methods based on energy, which were used by Petryk (1985, 1991), Schroder and Neff (2003), Wegner (2000, 2005, 2009) and Dargazany *et al.* (2012). Here, the necessary tool of description of mechanical properties of materials is the strain energy density function. It can be introduced in many different ways, for example as a direct function of invariants of the deformation state, see Wegner (1999, 2009) and Schroder and Neff (2003) or on the base of geometrical interpretation of the deformation process, see Wegner and Kurpisz (2009). Wegner and Kurpisz (2013), using a phenomenological approach and basing on the strain energy density function, investigated the damage process of a metal foam.

Material damage as a dissipative mechanism described in form of free energy per unit volume considered as a thermodynamic potential was taken into account by Cimetiere *et al.* (2005). The authors, in a major way, split the energy into two parts. The first is an elastic (reversible) part, whereas the second (dissipative) includes, among others, the hardening effect. Because each of these two parts depends on internal variables, so the internal variables influence the damage threshold.

Gajewska and Maciejewska (2005) investigated the influence of internal restrictions (connected with the type of the material, for example isotropic or anisotropic one) in form of limit conditions based on energy of anisotropic materials. Such conditions can be interpreted as different yield conditions. It was shown that as long as the energy scalar product was defined properly in the elastic range, the limit condition having the energy-based interpretation could be found. A much more interesting case takes place when we have nonlinear-elasticity, what implies the necessity of modification of the limit condition.

In the further part of the current paper, on the base of geometrical interpretation, the strain energy density function of a nonlinear material will be introduced and used for formulation of stability conditions due to the damage process, what is an alternative point of view to that presented by Gajewska and Maciejewska (2005).

2. Geometrical interpretation of the deformation state – the basic equation

To introduce a geometrical interpretation of the deformation process, let us take into account the following assumptions:

- The material is isotropic and nonlinear, so the mechanical properties are the same in all directions, but the relations between stress and strain can not be described in form of classic Hook's law (for linear materials).
- The loading process is static, which means that dynamic effects can be neglected.
- The dissipated part of energy includes thermal and plastic deformation energy.
- The material is under plane stress.
- The longitudinal deformation coefficient is a function of the deformation state due to nonlinear properties of the material.
- The principal stress and strain directions are the same due to material isotropy.

Every deformation process can be interpreted as a deformation path C, which is located in space of principal deformation state components. Every point of such path is one deformation state. So a change of deformation due to a change of external loads (change of principal stress components) implies a displacement along the path

$$C : \varepsilon_i = \varepsilon_i^k t \qquad \text{for} \quad i = 1,2,3 \tag{2.1}$$

where ε_i^k are the final deformation state components, and $t \in T = \langle 0,1 \rangle$ is a parameter.

The relation between stress and strain in every point of the deformation path takes the form of generalized Hook's law

$$\begin{aligned}
\varepsilon_1 &= \frac{\sigma_1(t)}{\widetilde{E}(\varepsilon_1)} - \widetilde{v}(\varepsilon_2)\frac{\sigma_2(t)}{\widetilde{E}(\varepsilon_2)} - \widetilde{v}(\varepsilon_3)\frac{\sigma_3(t)}{\widetilde{E}(\varepsilon_3)} \\
\varepsilon_2 &= \frac{\sigma_2(t)}{\widetilde{E}(\varepsilon_2)} - \widetilde{v}(\varepsilon_1)\frac{\sigma_1(t)}{\widetilde{E}(\varepsilon_1)} - \widetilde{v}(\varepsilon_3)\frac{\sigma_3(t)}{\widetilde{E}(\varepsilon_3)} \\
\varepsilon_3 &= \frac{\sigma_3(t)}{\widetilde{E}(\varepsilon_3)} - \widetilde{v}(\varepsilon_1)\frac{\sigma_1(t)}{\widetilde{E}(\varepsilon_1)} - \widetilde{v}(\varepsilon_2)\frac{\sigma_2(t)}{\widetilde{E}(\varepsilon_2)}
\end{aligned} \tag{2.2}$$

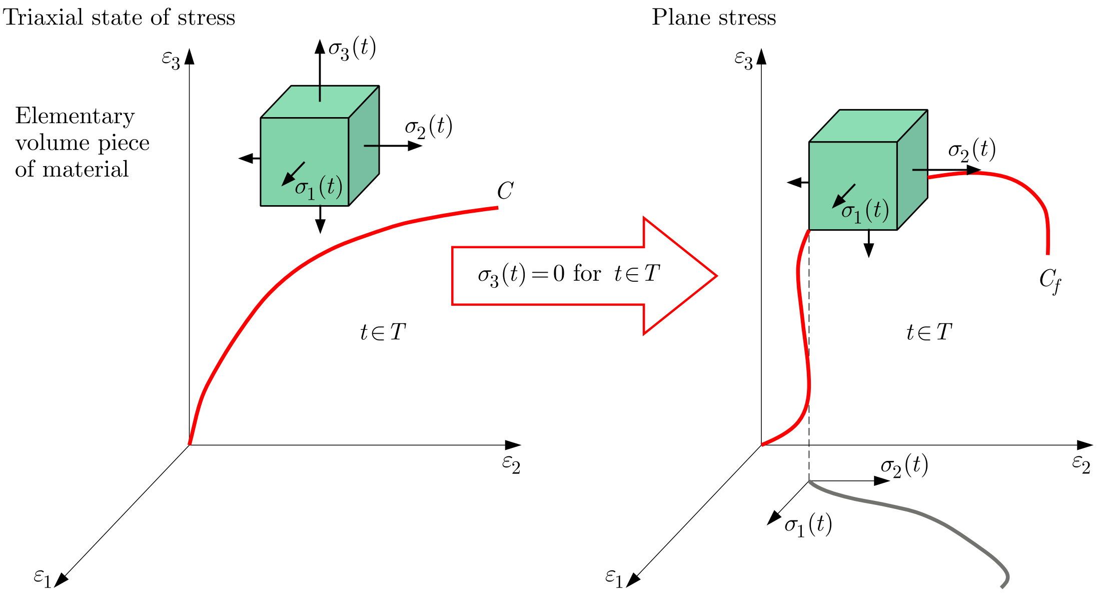

Fig. 1. Deformation paths for a triaxial and plane stress state in a material

where longitudinal and transversal deformation coefficients follow from experimental characteristics $\sigma(\varepsilon)$, $\varepsilon_t(\varepsilon)$ (ε_t is transversal deformation) obtained from a uniaxial tension test and take respectively the form

$$\widetilde{E}(\varepsilon) = \frac{\sigma}{\varepsilon} \qquad\qquad \widetilde{v}(\varepsilon) = -\frac{\varepsilon_t}{\varepsilon} \tag{2.3}$$

Because both stress and strain are changeable (along path C_f) in time $t \in \langle 0,1\rangle$ of the deformation process, then the density of deformation work from the initial state $\varepsilon_i = 0$ for $t = 0$ to the final state $\varepsilon_i = \varepsilon_i^k$ for $t = 1$ can be expressed as

$$W^C(\varepsilon_1^k, \varepsilon_2^k, \varepsilon_3^k) = \int\limits_{C_f} \sum_{i=1}^{3} \sigma_i \; d\varepsilon_i = \int\limits_0^1 \sum_{i=1}^{3} \sigma_i(t)\varepsilon_i'(t) \; dt \tag{2.4}$$

where on the basis of (2.2) for $i = 1,2,3$

$$\sigma_i(t) = \widetilde{E}(\varepsilon_i)\frac{\varepsilon_i \prod\limits_{l=1}^{3}[1+\widetilde{v}(\varepsilon_l)] + [1+\widetilde{v}(\varepsilon_i)]\sum\limits_{l=1}^{3}\widetilde{v}(\varepsilon_l)(\varepsilon_l-\varepsilon_i) + \frac{1+\widetilde{v}(\varepsilon_i)}{\widetilde{v}(\varepsilon_i)}\prod\limits_{l=1}^{3}\widetilde{v}(\varepsilon_l)\Big(\sum\limits_{l=1}^{3}\varepsilon_l - 3\varepsilon_i\Big)}{\prod\limits_{l=1}^{3}[1+\widetilde{v}(\varepsilon_l)] - [1+\widetilde{v}(\varepsilon_i)]^2\Big(\sum\limits_{l=1}^{3}\widetilde{v}(\varepsilon_l) - \widetilde{v}(\varepsilon_i) + \frac{2}{\widetilde{v}(\varepsilon_i)}\prod\limits_{l=1}^{3}\widetilde{v}(\varepsilon_l)\Big)} \tag{2.5}$$

In particular, if we assume the plane stress in plane $1O2$, then the stress components in the third direction are equal zero, and relations (2.4) and (2.5) simplify respectively to

$$W^C(\varepsilon_1^k, \varepsilon_2^k) = \int\limits_{C_f} \sum_{i=1}^{2} \sigma_i \; d\varepsilon_i = \int\limits_0^1 \sum_{i=1}^{2} \sigma_i(t)\varepsilon_i'(t) \; dt \tag{2.6}$$

and

$$\sigma_1(t) = \widetilde{E}(\varepsilon_1)\frac{\varepsilon_1 + \varepsilon_2\widetilde{v}(\varepsilon_2)}{1-\widetilde{v}(\varepsilon_1)\widetilde{v}(\varepsilon_2)} \qquad\qquad \sigma_2(t) = \widetilde{E}(\varepsilon_2)\frac{\varepsilon_2 + \varepsilon_1\widetilde{v}(\varepsilon_1)}{1-\widetilde{v}(\varepsilon_1)\widetilde{v}(\varepsilon_2)} \tag{2.7}$$

The strain component in the passive (third) direction takes the form

$$\varepsilon_3 = -\widetilde{v}(\varepsilon_1)\frac{\varepsilon_1 + \varepsilon_2\widetilde{v}(\varepsilon_2)}{1 - \widetilde{v}(\varepsilon_1)\widetilde{v}(\varepsilon_2)} - \widetilde{v}(\varepsilon_2)\frac{\varepsilon_2 + \varepsilon_1\widetilde{v}(\varepsilon_1)}{1 - \widetilde{v}(\varepsilon_1)\widetilde{v}(\varepsilon_2)} \tag{2.8}$$

In the geometrical interpretation, function (2.6) specifies the values of deformation work density (2.4), which are defined in the space of three-dimensional deformation to present a strain energy density distribution function plot (Fig. 5) along section surface (2.8).

3. Extraction of the volumetric part of energy

Let us consider purely volumetric deformations. Such type of deformations takes place if a material is under influence of hydrostatic pressure. The relation between deformation state components and stress state components according to hydrostatic pressure takes the form

$$\begin{aligned}
\varepsilon_1^V &= \frac{ks}{\widetilde{E}(\varepsilon_1^V)} - \widetilde{v}(\varepsilon_2^V)\frac{ks}{\widetilde{E}(\varepsilon_2^V)} - \widetilde{v}(\varepsilon_3^V)\frac{ks}{\widetilde{E}(\varepsilon_3^V)}\\
\varepsilon_2^V &= \frac{ks}{\widetilde{E}(\varepsilon_2^V)} - \widetilde{v}(\varepsilon_1^V)\frac{ks}{\widetilde{E}(\varepsilon_1^V)} - \widetilde{v}(\varepsilon_3^V)\frac{ks}{\widetilde{E}(\varepsilon_3^V)}\\
\varepsilon_3^V &= \frac{ks}{\widetilde{E}(\varepsilon_3^V)} - \widetilde{v}(\varepsilon_1^V)\frac{ks}{\widetilde{E}(\varepsilon_1^V)} - \widetilde{v}(\varepsilon_2^V)\frac{ks}{\widetilde{E}(\varepsilon_2^V)}
\end{aligned} \tag{3.1}$$

where $s \in T$ and hence, due to symmetry

$$\varepsilon_1^V = \varepsilon_2^V = \varepsilon_3^V = \varepsilon^V \tag{3.2}$$

where ε^V satisfies the equation

$$\varepsilon^V = \frac{1 - 2\widetilde{v}(\varepsilon^V)}{\widetilde{E}(\varepsilon^V)}ks = \beta(s) \tag{3.3}$$

and s is a non-dimensional parameter: the ratio of hydrostatic pressure to the basic value k, ks is the current value of hydrostatic pressure.

The above relation does not provide the information about the connection between the current value of deformation (point of path C) and its volumetric part (point of path C^V), see the picture below.

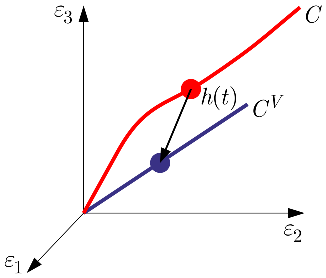

Fig. 2. Relation between the deformation path C and the path due to pure volumetric deformation C^V

To determinate this relation, we have to take into account two analytical descriptions of a volume change in an elementary piece of the material.

The first way

$$\frac{\Delta V}{V_0} = \prod_{i=1}^{3}[1 + \varepsilon_i(t)] - 1 \tag{3.4}$$

where $\varepsilon_1(t)$, $\varepsilon_2(t)$ and $\varepsilon_3(t)$ are deformation components of the path C.

The second way explores hydrostatic pressure

$$\varTheta = \frac{\varDelta V}{V_0} = \prod_{i=1}^{3}[1+\beta(s)] - 1 = [1+\beta(s)]^3 - 1 \tag{3.5}$$

where $\varTheta$ is the relative volume change.

After comparison of the right-hand sides of equations (3.4) and (3.5), we receive

$$s = h(t) = \beta^{-1}\Bigg(\sqrt[3]{\prod_{i=1}^{3}[1+\varepsilon_i(t)]} - 1\Bigg) \tag{3.6}$$

and after substitution into (3.3)

$$\varepsilon^V = \sqrt[3]{\prod_{i=1}^{3}[1+\varepsilon_i(t)]} - 1 \tag{3.7}$$

Hence, we can write that

$$W^{C^V} = \int\limits_{C^V} \sigma_1\, d\varepsilon_1^V + \int\limits_{C^V} \sigma_2\, d\varepsilon_2^V + \int\limits_{C^V} \sigma_3\, d\varepsilon_3^V = \sum_{i=1}^{3}\int\limits_{C^V} \sigma_i\, d\varepsilon^V \tag{3.8}$$

where σ_i $(i = 1, 2, 3)$ are solutions to the system of equations (2.2) given in form (2.5).

4. Stability conditions for the material under plane stress

The material is in the stable state of equilibrium if every change of the deformation state needs work to be done by external loads. So, in other words, we say about material stability when the strain energy density function is convex. In an analytical form, it can be written as

$$\delta^2 W^C = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 W^C}{\partial\varepsilon_i^k \partial\varepsilon_j^k}\delta\varepsilon_i^k \delta\varepsilon_j^k > 0 \tag{4.1}$$

where δ^2 denotes the second order variation of the function W^C.

In the case of plane stress, the deformation work W^C can be interpreted as a function of two deformation state components, however from the other side, the strain energy density depends on three deformation state components and the sign of its second order variation of the strain energy is strictly connected with the three variation increments of deformation state components. Hence, on the base of Sylvester's theorem, we receive

$$\begin{gathered}
\begin{vmatrix}
\dfrac{\partial^2 W^C}{\partial(\varepsilon_1^k)^2} & \dfrac{\partial^2 W^C}{\partial\varepsilon_1^k\partial\varepsilon_2^k} & \dfrac{\partial^2 W^C}{\partial\varepsilon_1^k\partial\varepsilon_3^k} \\
\dfrac{\partial^2 W^C}{\partial\varepsilon_1^k\partial\varepsilon_2^k} & \dfrac{\partial^2 W^C}{\partial(\varepsilon_2^k)^2} & \dfrac{\partial^2 W^C}{\partial\varepsilon_2^k\partial\varepsilon_3^k} \\
\dfrac{\partial^2 W^C}{\partial\varepsilon_1^k\partial\varepsilon_3^k} & \dfrac{\partial^2 W^C}{\partial\varepsilon_2^k\partial\varepsilon_3^k} & \dfrac{\partial^2 W^C}{\partial(\varepsilon_3^k)^2}
\end{vmatrix} > 0
\qquad
\begin{vmatrix}
\dfrac{\partial^2 W^C}{\partial(\varepsilon_1^k)^2} & \dfrac{\partial^2 W^C}{\partial\varepsilon_1^k\partial\varepsilon_2^k} \\
\dfrac{\partial^2 W^C}{\partial\varepsilon_1^k\partial\varepsilon_2^k} & \dfrac{\partial^2 W^C}{\partial(\varepsilon_2^k)^2}
\end{vmatrix} > 0
\\
\begin{vmatrix}
\dfrac{\partial^2 W^C}{\partial(\varepsilon_1^k)^2} & \dfrac{\partial^2 W^C}{\partial\varepsilon_1^k\partial\varepsilon_3^k} \\
\dfrac{\partial^2 W^C}{\partial\varepsilon_1^k\partial\varepsilon_3^k} & \dfrac{\partial^2 W^C}{\partial(\varepsilon_3^k)^2}
\end{vmatrix} > 0
\qquad
\begin{vmatrix}
\dfrac{\partial^2 W^C}{\partial(\varepsilon_2^k)^2} & \dfrac{\partial^2 W^C}{\partial\varepsilon_2^k\partial\varepsilon_3^k} \\
\dfrac{\partial^2 W^C}{\partial\varepsilon_2^k\partial\varepsilon_3^k} & \dfrac{\partial^2 W^C}{\partial(\varepsilon_3^k)^2}
\end{vmatrix} > 0
\\
\left|\dfrac{\partial^2 W^C}{\partial(\varepsilon_1^k)^2}\right| > 0
\qquad
\left|\dfrac{\partial^2 W^C}{\partial(\varepsilon_2^k)^2}\right| > 0
\qquad
\left|\dfrac{\partial^2 W^C}{\partial(\varepsilon_3^k)^2}\right| > 0
\end{gathered} \tag{4.2}$$

Inequality (4.1) implicates a system of six nonlinear inequalities (4.2) which allow us to draft the region of material stability. Because the plastic deformation leads to permanent loss of the element shape, so very important is the knowledge about material deformation due to plastic flow. This type of phenomenon takes place if during the deformation process, the volume change of an elementary piece of the material is unchanging.

On the basis of (A.15, see Appendix), the second order variation takes the form

$$\begin{aligned}\delta^2 W^{V=\text{const}} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial^2 W^C}{\partial\varepsilon_i^k \partial\varepsilon_j^k} - \frac{1}{3}\sum_{k=1}^{3}\frac{\sigma_k}{\sqrt[3]{A_1 A_2 A_3}}\frac{\partial A_i}{\partial \varepsilon_j^k}\right)\delta\varepsilon_i^k\delta\varepsilon_j^k \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} B_{ij}\delta\varepsilon_i^k\delta\varepsilon_j^k\end{aligned} \tag{4.3}$$

where A_1, A_2, A_3 are given as in Appendix (A.3), and σ_k are expressed by using (2.5).

Hence, stability assumption (4.3) takes respectively the form

$$\begin{aligned}&\begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33}\end{vmatrix} > 0 \qquad \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{vmatrix} > 0 \\ &\begin{vmatrix} B_{11} & B_{13} \\ B_{31} & B_{33}\end{vmatrix} > 0 \qquad \begin{vmatrix} B_{22} & B_{23} \\ B_{32} & B_{33}\end{vmatrix} > 0 \\ &|B_{11}| > 0 \qquad |B_{22}| > 0 \qquad |B_{33}| > 0\end{aligned} \tag{4.4}$$

In the case of a plane state of stress, so when $\sigma_3 = 0$, ε_3^k must to be replaced by relation (2.8).

5. Example

As an example of using theoretical investigations, aluminum in plain stress has been taken. The experimental plots of material characteristics are presented in Figs. 3 and 4.

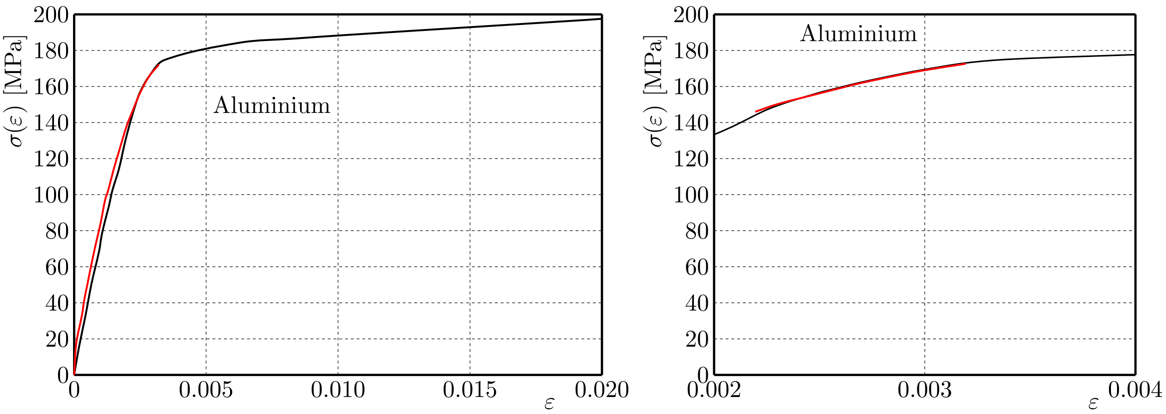

Fig. 3. Experimental relation between stress and strain (in a uniaxial tensile test) and its approximation in the nonlinear range (red line)

However, the precision of approximation of the experimental characteristic between stress and strain is not sufficient for very small (closed to zero) deformations, it is very accurate in the range $\varepsilon \in \langle 0.0023; 0.0033\rangle$, where there exists danger of the appearance of plastic flow. Because in the analytical assessment of the limit surface the second order partial derivatives of the strain

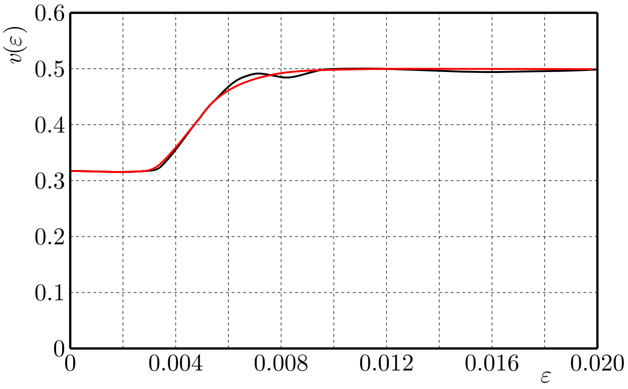

Fig. 4. Experimental plot of transversal deformation coefficient (relation between the ratio of transversal to longitudinal deformation and longitudinal deformation) and its approximation (red line)

energy density function are very important, then much more important is the accuracy of the approximation for $\varepsilon \in \langle 0.0023; 0.0033\rangle$ than for $\varepsilon \in \langle 0; 0.0023\rangle$.

Analytical approximations of the above characteristics (σ [MPa], $\widetilde{v}(\varepsilon)$ [MPa], $\widetilde{E}(\varepsilon)$ [MPa]) can be written respectively as

$$\begin{aligned} &\sigma = -11100000\varepsilon^2 + 86700\varepsilon + 9 \qquad \text{for} \quad \varepsilon \in \langle 0; 0.0033\rangle \\ &\widetilde{v}(\varepsilon) = \begin{cases} 0.317 & \text{for} \quad \varepsilon \in \langle 0; 0.0032\rangle \\ \dfrac{0.366}{\pi}\arctan\Big(2^{601\varepsilon}\dfrac{\varepsilon - 0.0032}{0.02-\varepsilon}\Big) + 0.317 & \text{for} \quad \varepsilon > 0.0032 \end{cases} \end{aligned} \tag{5.1}$$

Hence from $(2.3)_1$ and (5.1), we have

$$\widetilde{E}(\varepsilon) = -11100000\varepsilon + 86700 + \frac{9}{\varepsilon} \qquad \text{for} \quad \varepsilon \in \langle 0; 0.0033\rangle \tag{5.2}$$

The strain energy density function (W^C [MJ/m^3]) due to (2.5), (5.1) and (5.2) takes the form

$$\begin{aligned} W^C(\varepsilon_1^k, \varepsilon_2^k, \varepsilon_3^k) &= \int\limits_0^1 \sum_{i=1}^3 \sigma_i(t)\varepsilon_i'(t)\ dt = \int\limits_0^1 \sum_{i=1}^3 \sigma_i(t)\varepsilon_i^k\ dt \\ &= \frac{a_1(1+c_1) - 3a_1c_1}{3(1+c_1)(1-2c_1)} \sum_{i=1}^3 (\varepsilon_i^k)^3 + \frac{b_1(1+c_1) - 3b_1c_1}{2(1+c_1)(1-2c_1)} \sum_{i=1}^3 (\varepsilon_i^k)^2 \\ &+ \frac{a_1c_1}{3(1+c_1)(1-2c_1)} \sum_{i=1}^3 (\varepsilon_i^k)^2 \sum_{l=1}^3 \varepsilon_l^k + \frac{b_1c_1}{2(1+c_1)(1-2c_1)} \Big(\sum_{i=1}^3 \varepsilon_i^k\Big)^2 + \frac{1}{1-2c_1}\sum_{i=1}^3 \varepsilon_i^k \end{aligned} \tag{5.3}$$

where $a_1 = -11100000$, $b_1 = 86700$, $c_1 = 0.317$.

If we would like to represent and plot in an easy way the strain energy density function for a plain state of stress, then we have to substitute (on the basis of relation (2.8)) $\varepsilon_3^k = -[c/(1-c)](\varepsilon_1^k + \varepsilon_2^k)$ into relation (5.4). After transformations, we receive

$$\begin{aligned} W^C(\varepsilon_1^k, \varepsilon_2^k) &= -36815920[(\varepsilon_1^k)^3 + (\varepsilon_2^k)^3] + 431343.3[(\varepsilon_1^k)^2 + (\varepsilon_2^k)^2] \\ &+ 117.94(\varepsilon_1^k + \varepsilon_2^k) - 331343284[(\varepsilon_1^k)^2\varepsilon_2^k + \varepsilon_1^k(\varepsilon_2^k)^2] + 7764179.1\varepsilon_1^k\varepsilon_2^k \end{aligned} \tag{5.4}$$

for $\varepsilon_1^k \in \langle 0; 0.0033\rangle$, $\varepsilon_2^k \in \langle 0; 0.0033\rangle$.

If we apply relation (5.3) to stability assumptions (4.5) then we receive stability regions shown in Fig. 6.

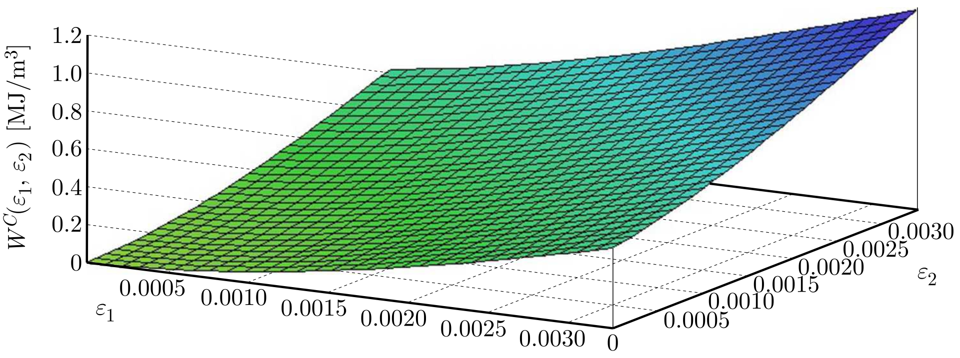

Fig. 5. The strain energy density function

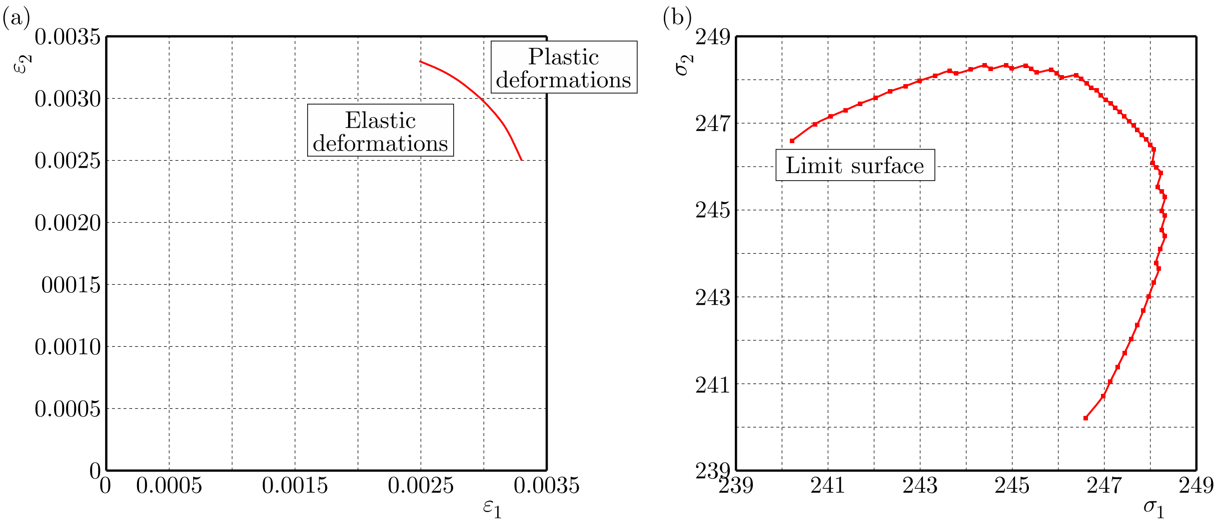

Fig. 6. Region of stability according to assumptions (4.5) in space of the deformation (a) and the stress (b) components

The above plots are obtained from the major form of stability assumptions (formulated for a three-axial state of stress), after substituting the relation $\varepsilon_3^k = -[c(1-c)](\varepsilon_1^k + \varepsilon_2^k)$, which is true in the case of plain stress.

6. Conclusions

- The strain energy density function is a sufficient tool in the description of mechanicals properties of nonlinear material and necessary for stability analysis.
- Extraction of the volumetric part of energy is possible due to hydrostatic interpretation.
- The constant volume assumption plays an important role in formulation of stability conditions due to plastic flow.
- The limit surfaces are convex (see. Figs. 6) and are comparable with the known limit surfaces for linear-elastic materials.

A. Appendix

The material is in the stable state of equilibrium if every change of current deformation state needs a work by external loads. So, if we take into account a very small fluctuation of the current deformation state in three principal directions then relation (4.1) has to be satisfied.

The plastic flow takes place in the case when an increment in volume equals zero, see (Wegner, 2000).

If l_1, l_2 and l_3 are dimensions of an elementary volume piece of the material in an unloaded state then the ratio of volume change to its virgin state can be written as

$$\theta = \frac{\varDelta V}{V_0} = \frac{\prod\limits_{i=1}^{3}(l_i + \varDelta l_i) - \prod\limits_{i=1}^{3} l_i}{\prod\limits_{i=1}^{3} l_i} = \prod_{i=1}^{3}(1+\varepsilon_i) - 1 \tag{A.1}$$

and hence

$$\delta\theta = \delta\Big[\prod_{i=1}^{3}(1+\varepsilon_i)\Big] = \sum_{i=1}^{3} A_i \delta\varepsilon_i \tag{A.2}$$

where

$$A_1 = (1+\varepsilon_2)(1+\varepsilon_3) \qquad A_2 = (1+\varepsilon_1)(1+\varepsilon_3) \qquad A_3 = (1+\varepsilon_1)(1+\varepsilon_2) \tag{A.3}$$

If we take into considerations purely volumetric deformations then $\varepsilon_i = \varepsilon^V$, which by substitution into (A.1) implies

$$\theta = (1+\varepsilon^V)^3 - 1 \;\Rightarrow\; \delta\theta = 3(1+\varepsilon^V)^2\delta\varepsilon^V \;\Rightarrow\; \delta\varepsilon^V = \frac{\delta\theta}{3(1+\varepsilon^V)^2} \tag{A.4}$$

The first order variation of the strain energy density function according to plastic flow can be divided as follows

$$\begin{aligned} \delta W^{(V=\mathrm{const})} &= \sigma_1\delta\varepsilon_1^{(V=\mathrm{const})} + \sigma_2\delta\varepsilon_2^{(V=\mathrm{const})} + \sigma_3\delta\varepsilon_3^{(V=\mathrm{const})} \\ \delta\varepsilon_i^{(V=\mathrm{const})} &= \delta\varepsilon_i - \delta\varepsilon^V \end{aligned} \tag{A.5}$$

In the case of the limit surface according to plastic flow, we do not observe a change in the volume, so

$$\delta W^{V=\mathrm{const}} = \delta W^C - \delta W^{C^V} = \delta W^s \tag{A.6}$$

where on the basis of (3.8), we have

$$\delta W^{C^V} = \sum_{i=1}^{3}\sigma_i\delta\varepsilon^V \tag{A.7}$$

Hence, taking into account (A.4)

$$\delta^2 W^{C^V} = \frac{1}{3}\sum_{i=1}^{3}\sigma_i\Big[\frac{\delta^2\theta}{(1+\varepsilon^V)^2} - \frac{2\delta\theta\delta\varepsilon^V}{(1+\varepsilon^V)^3}\Big] \tag{A.8}$$

In the absence of volumetric deformations

$$\delta\theta = \delta\varepsilon^V = 0 \tag{A.9}$$

we have

$$\delta^2 W^{C^V} = \frac{1}{3}\sum_{i=1}^{3}\sigma_i\frac{\delta^2\theta}{(1+\varepsilon^V)^2} \tag{A.10}$$

and after using (A.6)

$$\delta^2 W^{V=\text{const}} = \delta^2 W^C - \delta^2 W^{C^V} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial^2 W^C}{\partial\varepsilon_i \partial\varepsilon_j}\delta\varepsilon_i\delta\varepsilon_j - \frac{1}{3}\sum_{i=1}^{3}\sigma_i \frac{\delta^2\theta}{(1+\varepsilon^V)^2} \tag{A.11}$$

which on the basis of (A.2) gives

$$\delta^2 W^{V=\text{const}} = \sum_{i=1}^{3}\sum_{j=1}^{3}\Big(\frac{\partial^2 W^C}{\partial\varepsilon_i \partial\varepsilon_j} - \frac{1}{3}\sum_{k=1}^{3}\frac{\sigma_k}{(1+\varepsilon^V)^2}\frac{\partial A_i}{\partial\varepsilon_j}\Big)\delta\varepsilon_i\delta\varepsilon_j \tag{A.12}$$

On the grounds of (A.4)

$$\begin{gathered} 1+\theta = (1+\varepsilon^V)^3 \;\Rightarrow\; 1+\varepsilon^V = \sqrt[3]{1+\theta} \;\Rightarrow\; 1+\varepsilon^V = \sqrt[3]{\prod_{i=1}^{3}(1+\varepsilon_i)} \\ \Rightarrow\; (1+\varepsilon^V)^2 = \sqrt[3]{A_1 A_2 A_3} \end{gathered} \tag{A.13}$$

The above relation enables us to write (A.12 in an equivalent form

$$\delta^2 W^{V=\text{const}} = \sum_{i=1}^{3}\sum_{j=1}^{3}\Big(\frac{\partial^2 W^C}{\partial\varepsilon_i \partial\varepsilon_j} - \frac{1}{3}\sum_{k=1}^{3}\frac{\sigma_k}{\sqrt[3]{A_1 A_2 A_3}}\frac{\partial A_i}{\partial\varepsilon_j}\Big)\delta\varepsilon_i\delta\varepsilon_j \tag{A.14}$$

Acknowledgments
This study was supported by DS: 02/21/DSPB/3452.

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Manuscript received November 21, 2014; accepted for print June 15, 2016