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FRACTIONAL HEAT CONDUCTION IN A SPHERE UNDER MATHEMATICAL AND PHYSICAL ROBIN CONDITIONS

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In this paper, the effect of a fractional order of time-derivatives occurring in fractional heat conduction models on the temperature distribution in a composite sphere is investigated. The research concerns heat conduction in a sphere consisting of a solid sphere and a spherical layer which are in perfect thermal contact. The solution of the problem with a classical Robin boundary condition and continuity conditions at the interface in an analytical form has been derived. The fractional heat conduction is governed by the heat conduction equation with the Caputo time-derivative, a Robin boundary condition and a heat flux continuity condition with the Riemann-Liouville derivative. The solution of the problem of non-local heat conduction by using the Laplace transform technique has been determined, and the temperature distribution in the sphere by using a method of numerical inversion of the Laplace transforms has been obtained.

Keywords: heat conduction, fractional heat equation, Robin boundary condition

1. Introduction

The classical heat conduction model based on the Fourier law has a non-physical property that the heat propagates with an infinite speed (Özişik, 1993). This property is a consequence of the dependence between the heat flux vector and the temperature gradient which is established by the Fourier law. This disadvantage does not appear when the non-local time dependence between the flux vector and the temperature gradient is assumed (Povstenko, 2014; Sur and Kanoria, 2014). This assumption leads to a differential equation and/or boundary conditions with derivatives of a non-integer order. The properties of fractional derivatives and different analytical methods to solve fractional differential equations are presented in (Atanacković *et al.*, 2014; Klimek, 2009; Leszczyński, 2011; Magin, 2006; Mainardi, 2010; Povstenko, 2015). Approximate numerical methods were applied to solving fractional initial-boundary problems in numerous papers, for example in (Blaszczyk and Ciesielski, 2017; Ciesielski and Błaszczyk, 2013; Dimitrov, 2014).

The heat conduction modelled by using the fractional order derivative is the subject of many papers. A mathematical model of one-dimensional heat conduction in a slab was proposed in paper (Žecová and Terpák, 2015). The Grünwald-Letnikov derivative with respect to a time variable was used. A solution to the problem of fractional heat conduction in a two-layered slab with the Caputo time-derivative in the heat conduction equation was presented in (Kukla and Siedlecka, 2015). Heat transfer for non-contacting face seals described by the time-fractional heat conduction equation in the cylindrical coordinate system was considered in (Blasiak, 2016). The fractional model of thermal energy transport in rigid bodies was derived in (Raslan, 2016). The effect of the order of the Marchand-type derivative in the heat transfer equation on the temperature distribution in a rigid conductor was numerically investigated. An application of the fractional order theory to a problem of thermal stress distribution in a spherical shell was studied in (Zingales, 2014). In the paper by Atangana and Bildik (2013), the time fractional calculus was employed in the mathematical model of groundwater flow. Applications of fractional order systems to an ultracapacitor and beam heating problems were presented in (Dzieliński *et al.*, 2010). An application of fractional calculus in continuum mechanics to a problem of linear elasticity under small deformation was shown in (Sumelka and Blaszczyk, 2014). Some applications of the fractional calculus were also discussed in the papers (Abbas, 2012; Dalir and Bashour, 2010; Rahimy, 2010).

Solutions to time-fractional heat conduction problems in a spherical coordinate system are presented in many papers. In the paper by Ning and Jiang (2011), for the problem of fractional heat conduction in a sphere, the method of the Laplace transform and the variable separation were used. An analytical solution to the problem of the time-fractional radial heat conduction in a multilayered sphere under the Robin boundary condition was presented by Kukla and Siedlecka (2017). Fundamental solutions to the Cauchy problem and to the source problem of the heat conduction fractional equation in a spherical coordinate system in an analytical form were derived by article Povstenko and Klekot (2017).

The fractional heat conduction equation is complemented by initial and boundary conditions. Mathematical and physical formulations of the initial and boundary conditions can be considered in fractional heat conduction models (Povstenko, 2013). The mathematical formulations of Dirichlet, Neumann and Robin boundary conditions are the same as these in the classical theory of heat conduction. Also, the physical Dirichlet condition has the same form as the classical boundary condition of the first kind, while the physical Neumann and the physical Robin boundary conditions contain the fractional time-derivative. If two solids are in perfect thermal contact, the physical formulation of the condition of heat flux equality through the contact surface also contain the fractional time-derivative (Povstenko, 2013).

The solution to the problem of linear fractional heat conduction in a sphere under mathematical boundary conditions can be determined in an analytical form. However, in solving such problems of heat conduction under physical Neumann or Robin boundary conditions, an approximate methods must be used. Application of the Laplace transform method to a linear problem allows one to obtain a solution in the Laplace domain. For the fractional heat conduction problems under physical Neumann or Robin boundary conditions and physical continuity conditions, the inverse Laplace transform in an analytical form can not be determined. The solution to the problem is obtained by applying numerical inversion of the Laplace transform. The methods for numerical inversion of the Laplace transform, which are used in the classical problems, can be also applied to the Laplace transform obtained in fractional analysis. A review of the methods to numerical inversion of the Laplace transform was presented by Kuhlman (2013). An application of selected methods to determine the inverse Laplace transform in fractional calculus were presented in (Brzeziński and Ostalczyk, 2016; Sheng *et al.*, 2011). Modification of the method introduced by Gaver (1966) was presented in (Abate and Valkó, 2004; Valkó and Abate, 2004).

In this paper, the fractional heat conduction problem in a solid sphere under the mathematical and physical boundary condition is studied. The considered sphere consists of an inner sphere and a spherical layer. We assume perfect thermal contact of the inner sphere and the spherical layer which is modeled by mathematical or physical conditions. The exact solution of the problem for the mathematical boundary condition and the solution in the Laplace domain for the physical formulation of boundary and continuity conditions are presented. The effect of the order of the Riemman-Liouville derivative in the Robin physical condition and in the contact condition at the interface on the temperature distribution in the sphere has been numerically investigated.

2. Formulation of the problem

We consider the problem of heat conduction in a sphere which consists of a solid sphere occupying the region $0 \leqslant r \leqslant r_1$ and a spherical layer defined by $r_1 \leqslant r \leqslant b$, in the spherical coordinates system. The time-fractional heat conduction in the inner sphere ($i = 1$) and in the spherical layer ($i = 2$) is governed by the following equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial T_i}{\partial r}\Big) = \frac{1}{a_i}\frac{\partial^{\alpha_i} T_i}{\partial t^{\alpha_i}} \qquad i = 1, 2 \tag{2.1}$$

where a_i is the thermal diffusivity, λ_i is the thermal conductivity and α_i denotes the fractional order of the left Caputo derivative with respect to time t. The Caputo derivative is defined by (Podlubny, 1999)

$${}_a^C D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)}\int\limits_a^t (t-\tau)^{m-\alpha-1}\frac{d^m f(\tau)}{d\tau^m}\,d\tau \qquad m-1 < \alpha < m \tag{2.2}$$

We consider the case of $a = 0$ and $\alpha \in (0, 1]$. Note, that the thermal diffusivity coefficient can be interpreted as a measure of the distance on which the thermal front propagates in a medium at the given time. The thermal conductivity is a measure of the ability of the medium to transfer heat.

The condition at the centre of the sphere, the continuity conditions at the interface, the Robin boundary condition on the outer surface and the initial condition are (Povstenko, 2013a,b)

$$|T(0,t)| < \infty \qquad T_1(r_1,t) = T_2(r_1,t) \tag{2.3}$$

$$\begin{aligned} &\lambda_1 D_{RL}^{1-\beta_1}\frac{\partial T_1}{\partial r}(r_1,t) = \lambda_2 D_{RL}^{1-\beta_2}\frac{\partial T_2}{\partial r}(r_1,t) \\ &\lambda_2 D_{RL}^{1-\beta_2}\frac{\partial T_2}{\partial r}(b,t) = a_\infty(T_\infty(t) - T_2(b,t)) \end{aligned} \tag{2.4}$$

$$T(r,0) = F_i(r) \tag{2.5}$$

where a_∞ is the outer heat transfer coefficient and T_∞ is the ambient temperature. The left Riemann-Liouville fractional derivative $D_{RL}^{1-\beta}$ which occurs in equations (2.4) is defined by (Diethelm, 2010)

$$D_{RL}^\beta f(t) = \frac{d}{dt}\Big(\frac{1}{\Gamma(1-\beta)}\int\limits_0^t \frac{f(\tau)}{(t-\tau)^\beta}\,d\tau\Big) \qquad 0 < \beta \leqslant 1 \tag{2.6}$$

Conditions (2.4) for $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$ for $\alpha_1, \alpha_2 \in (0, 1)$ are called the physical conditions (Rahimt, 2010; Raslan, 2016). If $\beta_1 = \beta_2 = 1$, the conditions are called the mathematical conditions. In this case, the D_{RL}^0 means an identity operator and can be omitted in equations (2.4).

3. Solution to the problem

The fractional heat conduction problem defined by equations (2.1) and (2.3)-(2.5) can be transformed to a new problem for functions $U_i(r,t)$ by using the formula

$$U_i(r,t) = r\,(T_i(r,t) - T_\infty(t)) \qquad i = 1, 2 \tag{3.1}$$

Taking into account relationship (3.1) in equation (2.1) and conditions (2.3)-(2.5), we obtain a formulation of the initial-boundary problem in the form

$$a_i \frac{\partial^2 U_i(r,t)}{\partial r^2} = \frac{\partial^{\alpha_i} U_i(r,t)}{\partial t^{\alpha_i}} + r\frac{d^{\alpha_i} T_\infty(t)}{dt^{\alpha_i}} \qquad i = 1,2 \tag{3.2}$$

$$U_1(0,t) = 0 \qquad U_1(r_1,t) = U_2(r_1,t) \tag{3.3}$$

$$\begin{aligned} &\lambda_1 D_{RL}^{1-\beta_1}\Big(\frac{\partial U_1(r_1,t)}{\partial r} - \frac{1}{r_1}U_1(r_1,t)\Big) = \lambda_2 D_{RL}^{1-\beta_2}\Big(\frac{\partial U_2(r_1,t)}{\partial r} - \frac{1}{r_1}U_2(r_1,t)\Big) \\ &\lambda_2 D_{RL}^{1-\beta_2}\Big(\frac{\partial U_2}{\partial r}(b,t) - \frac{1}{b}U_2(b,t)\Big) = -a_\infty U_2(b,t) \end{aligned} \tag{3.4}$$

$$U_i(r,0) = r\,(F_i(r) - T_\infty(0)) \qquad i = 1,2 \tag{3.5}$$

The solution to initial-boundary problem (3.2)-(3.5) for $\beta_1 = \beta_2 = 1$ (mathematical formulation) and for $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$ (physical formulation) will be presented below.

3.1. Mathematical conditions

An analytical solution to time-fractional heat conduction problem (3.2)-(3.5) under mathematical conditions (3.4) for $\alpha_1 = \alpha_2 = \alpha$ will be determined by using the method of separation of variables. As a result, we find a solution to the problem in the form of a series

$$U_i(r,t) = \sum_{k=1}^{\infty} \varLambda_k(t)\varPhi_{i,k}(r) \qquad i = 1,2 \tag{3.6}$$

The functions $\varPhi_{1,k}(r)$ and $\varPhi_{2,k}(r)$ for $k = 1,2,\ldots$ are obtained as a solution to the corresponding eigenvalue problem

$$\frac{d^2\varPhi_{i,k}(r)}{dr^2} + \frac{\gamma_k^2}{a_i}\varPhi_{i,k}(r) = 0 \qquad i = 1,2 \tag{3.7}$$

$$\varPhi_{1,k}(0) = 0 \qquad \varPhi_{1,k}(r_1) = \varPhi_{2,k}(r_1) \tag{3.8}$$

$$\lambda_1\frac{d\varPhi_1(r_1)}{dr} + \frac{1}{r_1}(\lambda_2 - \lambda_1)\varPhi_1(r_1) = \lambda_2\frac{d\varPhi_2(r_1)}{dr} \qquad \frac{d\varPhi_2(b)}{dr} = \Big(\frac{1}{b} - \frac{a_\infty}{\lambda_2}\Big)\varPhi_2(b) \tag{3.9}$$

The eigenfunctions $\varPhi_{i,k}(r)$ are given by

$$\varPhi_{1,k}(r) = B_{1,k}\sin\mu_{1,k}r \qquad \varPhi_{2,k}(r) = A_{2,k}\cos\mu_{2,k}(r - r_1) + B_{2,k}\sin\mu_{2,k}(r - r_1) \tag{3.10}$$

where $\mu_{i,k} = \gamma_k/\sqrt{a_i}$ and γ_k are the roots of the eigenvalue equation

$$M_1\lambda_2\mu_1\sin\mu_1 r_1 + M_2M_3 = 0 \tag{3.11}$$

where

$$\begin{aligned} M_1 &= \Big(\frac{a_\infty}{\lambda_2} - \frac{1}{b}\Big)\cos\mu_2(b - r_1) - \mu_2\sin\mu_2(b - r_1) \\ M_2 &= \Big(\frac{a_\infty}{\lambda_2} - \frac{1}{b}\Big)\sin\mu_2(b - r_1) + \mu_2\cos\mu_2(b - r_1) \\ M_3 &= \lambda_1\mu_1\cos\mu_1 r_1 + \frac{\lambda_2 - \lambda_1}{r_1}\sin\mu_1 r_1 \end{aligned}$$

The coefficients $B_{1,k}$, $A_{2,k}$ and $B_{2,k}$ are determined by using continuity and boundary conditions (3.8) and (3.9). Assuming $B_{1,k} = 1$, we obtain $A_{2,k} = \sin\mu_{1,k}r_1$ and $B_{2,k} = M_3/\lambda_2\mu_{2,k}$.

The function $\varLambda_k(t)$, occurring in equation (3.6), is a solution to the fractional initial problem which is obtained by using the orthogonality condition in the form

$$\frac{\lambda_1}{a_1}\int\limits_0^{r_1} \varPhi_{1,k}(r)\varPhi_{1,k'}(r)\,dr + \frac{\lambda_2}{a_2}\int\limits_{r_1}^{b} \varPhi_{2,k}(r)\varPhi_{2,k'}(r)\,dr = \begin{cases} 0 & \text{for} \quad k' \neq k \\ N_k & \text{for} \quad k' = k \end{cases} \tag{3.12}$$

Assuming $F_i(r) = T_{init} =$ const for $i = 1, 2$ and condition (3.12) in equation (3.2) and (3.5), the initial problem is obtained

$$\begin{aligned} &\frac{d^\alpha \varLambda_k(t)}{dt^\alpha} + \gamma_k^2 \varLambda_k(t) = 0 \\ &\varLambda_k(0) = \frac{T_{init} - T_\infty}{N_k^r}\Big(\frac{\lambda_1}{a_1}\int\limits_0^{r_1} r\varPhi_{1,k}(r)\,dr + \frac{\lambda_2}{a_2}\int\limits_{r_1}^{b} r\varPhi_{2,k}(r)\,dr\Big) \end{aligned} \tag{3.13}$$

A solution to problem (3.13) is given by (Diethelm, 2010)

$$\varLambda_k(t) = \frac{T_{init} - T_\infty}{N_k^r} E_\alpha(-\gamma_k^2 t^\alpha)\Big(\frac{\lambda_1}{a_1}\int\limits_0^{r_1} r\varPhi_{1,k}(r)\,dr + \frac{\lambda_2}{a_2}\int\limits_{r_1}^{b} r\varPhi_{2,k}(r)\,dr\Big) \tag{3.14}$$

where $E_\alpha(z)$ is the Mittag-Leffler function (Kilbas *et al.*, 2006)

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\varGamma(\alpha k + 1)} \tag{3.15}$$

Finally, the functions $T_i(r,t)$ are given by equations (3.1), (3.6), (3.10) and (3.14). Assuming that the following conditions are fulfilled: $a_1 = a_2 = a$, $\lambda_1 = \lambda_2 = \lambda$, $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = 1$, we obtain the temperature $T(r,t)$ in the homogeneous sphere

$$T(r,t) = T_\infty + \frac{4(T_{init} - T_\infty)}{r} \sum_{k=1}^{\infty} \frac{b\mu_k \cos b\mu_k - \sin b\mu_k}{\mu_k(\sin 2b\mu_k - 2b\mu_k)} E_\alpha(-\gamma_k^2 t^\alpha) \sin \mu_k r \tag{3.16}$$

In this case, $\mu_k = \gamma_k/\sqrt{a}$ and γ_k are the roots of the equation

$$\Big(1 - \frac{ba_\infty}{\lambda}\Big)\sin b\mu - b\mu \cos b\mu = 0 \tag{3.17}$$

3.2. Physical conditions

We obtain a solution to problem (3.2)-(3.5) under physical boundary and continuity conditions ($\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$ in equations (3.4) and (3.5)) by using the Laplace transform method. The Laplace transform $\overline{f}(s)$ of a function $f(t)$ is defined by

$$\overline{f}(s) = \int\limits_0^{\infty} f(t)\mathrm{e}^{-st}\,dt \tag{3.18}$$

where s is a complex parameter. Using the properties of the Laplace transform, equations (3.2)-(3.4) can be rewritten in the Laplace domain as

$$\frac{d^2\overline{U}_i}{dr^2} - \frac{s^{\alpha_i}}{a_i}\overline{U}_i(r,s) = \frac{rs^{\alpha_i}}{a_i}\Big(\overline{T}_\infty(s) - \frac{F_i(r)}{s}\Big) \tag{3.19}$$

$$\overline{U}_1(0,s) = 0 \qquad\qquad \overline{U}_1(r_1,s) = \overline{U}_2(r_1,s) \tag{3.20}$$

$$
\begin{aligned}
&\lambda_1 s^{1-\alpha_1}\Big(\frac{d\overline{U}_1(r_1,s)}{dr} - \frac{1}{r_1}\overline{U}_1(r_1,s)\Big) = \lambda_2 s^{1-\alpha_2}\Big(\frac{d\overline{U}_2(r_1,s)}{dr} - \frac{1}{r_1}\overline{U}_2(r_1,s)\Big)\\
&\lambda_2 s^{1-\alpha_2}\Big(\frac{d\overline{U}_2(b,s)}{dr} - \frac{1}{b}\overline{U}_2(b,s)\Big) = -a_\infty \overline{U}_2(b,s)
\end{aligned}
\tag{3.21}
$$

The general solution to equation (3.19) for $i = 1,2$ has the form

$$
\begin{aligned}
\overline{U}_1(r,s) &= B_1 \sinh S_1 r + \frac{1}{S_1}\int\limits_0^r P_1(u)\sinh S_1(r-u)\,du\\
\overline{U}_2(r,s) &= A_2 \cosh S_2(r-r_1) + B_2\sinh S_2(r-r_1) + \frac{1}{S_2}\int\limits_{r_1}^r P_2(u)\sinh S_2(r-u)\,du
\end{aligned}
\tag{3.22}
$$

where

$$
S_i = \frac{s^{\alpha_i/2}}{\sqrt{a_i}} \qquad\qquad P_i(r) = \frac{rs^{\alpha_i}}{a_i}\Big(\overline{T}_\infty(s) - \frac{F_i(r)}{s}\Big)
$$

The constants B_1, A_2 and B_2 are determined by using conditions $(3.20)_2$ and (3.21). After some transformations, the functions $\overline{U}_1(r,s)$ and $\overline{U}_2(r,s)$ can be written as

$$
\overline{U}_1(r,s) = \widetilde{B}_1 \sinh S_1 r \qquad\qquad \overline{U}_2(r,s) = \widetilde{A}_2\cosh S_2(r-r_1) + \widetilde{B}_2 \sinh S_2(r-r_1) \tag{3.23}
$$

where

$$
\begin{aligned}
\widetilde{B}_1 &= -s^{\alpha_1-2}\frac{a_\infty}{\lambda_1 d}S_2 b^2 \qquad\qquad \widetilde{A}_2 = -s^{\alpha_1-2}\frac{a_\infty b}{\lambda_1 d}S_2 b\sinh S_1 r_1\\
\widetilde{B}_2 &= s^{\alpha_2-2}\frac{b}{r_1}\frac{a_\infty b}{\lambda_2 d}\Big[\Big(1 - s^{\alpha_1-\alpha_2}\frac{\lambda_2}{\lambda_1}\Big)\sinh S_1 r_1 - S_1 r_1\cosh S_1 r_1\Big]\\
d &= s^{\alpha_1-\alpha_2}\frac{\lambda_2}{\lambda_1}S_2 r_1 \sinh S_1 r_1[w\cosh S_2(b-r_1) + S_2 b\sinh S_2(b-r_1)]\\
&\quad + \Big]S_1 r_1\cosh S_1 r_1 - \Big(1 - s^{\alpha_1-\alpha_2}\frac{\lambda_2}{\lambda_1}\Big)\sinh S_1 r_1\Big][S_2 b\cosh S_2(b-r_1) + w\sinh S_2(b-r_1)]\\
w &= \frac{a_\infty b}{\lambda_2 s^{1-\alpha_2}} - 1
\end{aligned}
$$

Assuming $F_i(r) = T_{init} =$ const for $i = 1,2$, the temperature distribution in the sphere is given by

$$
T_i(r,t) = T_\infty + (T_{init} - T_\infty)\frac{r_1}{r}L^{-1}[\overline{U}_i(r,s)] \tag{3.24}
$$

For the homogeneous sphere, the following conditions are fulfilled $a_1 = a_2 = a$, $\lambda_1 = \lambda_2 = \lambda$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $S_1 = S_2 = S$. In this case, the function $T_i(r,t) = T(r,t)$ has the form

$$
T(r,t) = T_\infty + (T_{init} - T_\infty)\frac{b}{r}L^{-1}[\overline{U}(r,s)] \tag{3.25}
$$

where

$$
\overline{U}(r,s) = -\frac{1+w}{s(Sb\cosh Sb + w\sinh Sb)}\sinh Sr
$$

The inverse of the Laplace transform of the functions $\overline{U}_1(r,s)$ and $\overline{U}_2(r,s)$ are numerically determined. The calculation has been performed by the Gaver method using the sequence of functionals presented in Gaver (1966) and Valkó and Abate (2004). Applying this method, the approximate values of the original function $U_i(r,t) = L^{-1}[\overline{U}_i(r,s)]$ are determined using the formula

$$U_i(r,t) \simeq n\tau \binom{2n}{n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \overline{U}_i(r,(n+i)\tau) \tag{3.26}$$

where $\tau = (\ln 2)/t$ and n is a fixed positive integer number.

The functions $T_i(r,t)$ and $T(r,t)$ obtained for the mathematical and physical conditions will serve for investigation of the influence of the orders of the Caputo and Riemann-Liouville derivatives occurring in the heat conduction models on the temperature distribution in the sphere.

4. Results of numerical calculations

The effect of the order of the fractional derivative in the heat conduction equation on the temperature distribution in the sphere has been numerically investigated. The results for the mathematical boundary condition obtained by using numerical inversion of the Laplace transforms has been compared with the exact solution. The computations were performed for the homogeneous sphere (Sphere A) and for the sphere consisting of a solid sphere and a spherical layer (Sphere B). The radius of both Spheres was $b = 1.0\,\mathrm{m}$ and the interface in Sphere B was at $\hat{r}_1 = r_1/b = 0.9$. The thermal diffusivity $a = 3.352 \cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}^\alpha$ and the thermal conductivity $\lambda = 16\,\mathrm{W/(m{\cdot}K)}$ were assumed for Sphere A. The thermal diffusivities $a_1 = 2.3 \cdot 10^{-5}\,\mathrm{m}^2/\mathrm{s}^\alpha$, $a_2 = 3.352 \cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}^\alpha$ and the thermal conductivities $\lambda_1 = 80\,\mathrm{W/(m{\cdot}K)}$, $\lambda_2 = 16\,\mathrm{W/(m{\cdot}K)}$ were assumed for Sphere B. Subscript 1 was used for the inner sphere and subscript 2 – for the spherical layer of Sphere B. For both Spheres, the outer heat transfer coefficient was $a_\infty = 200\,\mathrm{W/(m^2{\cdot}K)}$, the ambient temperature was $T_\infty = 100°\mathrm{C}$ and the initial temperature was assumed as $T_{init} = 25°\mathrm{C}$.

In Table 1, the non-dimensional temperature $\widehat{T} = T/T_{init}$ in Sphere A for different orders of the Caputo derivative α at the reference time $\widehat{t} = tb^2/a = 1.0$ is presented. The calculation has been performed for the mathematical Robin boundary condition, i.e. for $\beta = 1.0$. The results were obtained by using the exact solution, Eq. (3.16), and by the Gaver method of numerical inversion of the Laplace transforms, Eq. (3.26), and using relationship (3.1). A similar comparison of numerically obtained non-dimensional temperatures have been performed for Sphere B. The results are presented in Table 2. The relative error evaluated on the basis of the results given in Tables 1 and 2 fulfils the condition: $|Exact - NILT|/Exact < 3.6 \cdot 10^{-5}$. The good accordance of the results obtained for mathematical formulation of the boundary and continuity condition allows one to use the NILT method to the heat conduction problem under physical formulation of the boundary and continuity condition.

Table 1. Non-dimensional temperature $\widehat{T}(\hat{r},\hat{t})$ for $\hat{t} = 1.0$, computed by using the exact solution and by using numerical inversion of the Laplace transform (NILT) for Sphere A

$\hat{r}$	$\alpha = 0.8$		$\alpha = 0.9$		$\alpha = 1.0$	
	Exact	NILT	Exact	NILT	Exact	NILT
0	1.12416	1.12412	2.41882	2.41882	3.91149	3.91145
0.25	1.19635	1.19634	2.52818	2.52818	3.91902	3.91900
0.50	1.49058	1.49058	2.84104	2.84103	3.93932	3.93932
0.75	2.21186	2.21186	3.30678	3.30678	3.96637	3.96629
1.00	3.50777	3.50776	3.83398	3.83398	3.99254	3.99251

Table 2. Non-dimensional temperature $\widehat{T}(\hat{r},\hat{t})$ for $\hat{t}=1.0$, computed by using the exact solution and by using numerical inversion of the Laplace transform (NILT) for Sphere B

$\hat{r}$	$\alpha=0.8$		$\alpha=0.9$		$\alpha=1.0$	
	Exact	NILT	Exact	NILT	Exact	NILT
0	2.35412	2.35407	3.66988	3.66988	3.99982	3.99969
0.25	2.39527	2.39526	3.67879	3.67880	3.99983	3.99969
0.50	2.51791	2.51791	3.70498	3.70498	3.99984	3.99971
0.75	2.71870	2.71870	3.74679	3.74679	3.99987	3.99974
1.00	3.48174	3.48172	3.89885	3.89885	3.99995	3.99988

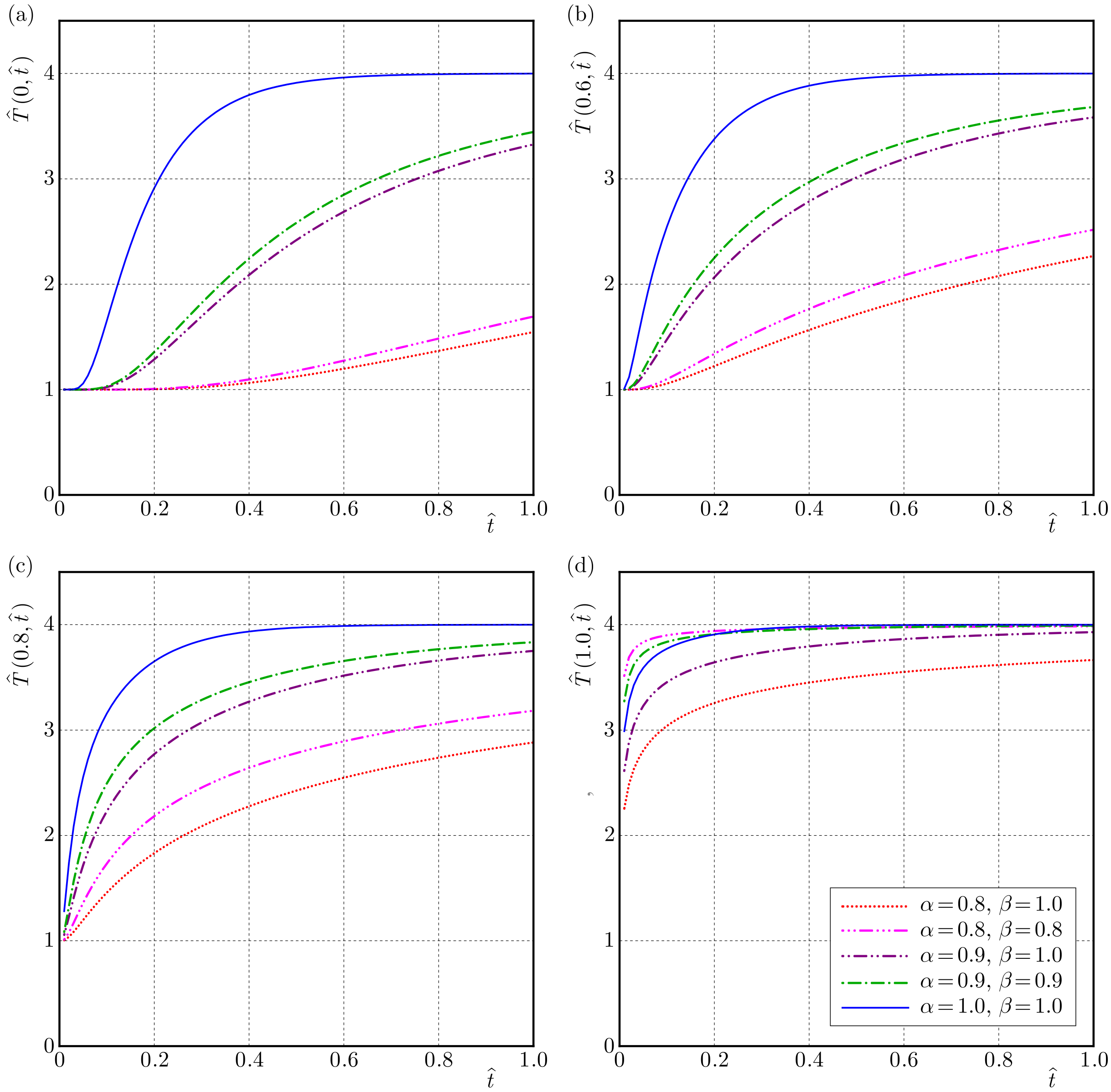

Fig. 1. Non-dimensional temperature $\widehat{T}(\hat{r},\hat{t})$ as a function of time $\hat{t}$ in Sphere A for various values of fractional derivatives α and β: (a) $\hat{r}=0$, (b) $\hat{r}=0.6$, (c) $\hat{r}=0.8$, (d) $\hat{r}=1.0$

The non-dimensional temperatures $\widehat{T}$ as functions of the time $\hat{t}$ for various radial coordinates are presented in Fig. 1. The pairs of curves obtained for mathematical and physical formulations of the Robin condition show the effect of the order of the Riemann-Liouville derivative occurring in the physical boundary condition on the temperature in the sphere. Significant differences can be observed in the temperatures obtained for the classical heat conduction model ($\alpha = \beta = 1$) and fractional models ($\alpha = 0.8$ and $\alpha = 0.9$), particularly in the inner points of the sphere.

The curves presented in Fig. 2 represent the non-dimensional temperatures $\widehat{T}$ as functions of the reference time $\hat{t}$ for Sphere B. In numerical calculations with the mathematical conditions (MC) the following values have been assumed $\alpha_1 = \alpha_2 = \alpha = 0.8,\ 0.9,\ 1.0$ and $\beta_1 = \beta_2 = 1.0$. The numerical calculations to the problem with the physical conditions (PC) have been carried out for: $\alpha_1 = \beta_1 = 0.9$, $\alpha_2 = \alpha = 0.8,\ 0.9,\ 1.0$ and $\beta_2 = \beta = \alpha_2$. A higher temperature is observed for the heat conduction with the physical boundary and continuity conditions than in the model with the mathematical formulation of these conditions. A significant effect on the temperature distribution in the sphere has the order of the Caputo derivative in the heat conduction model.

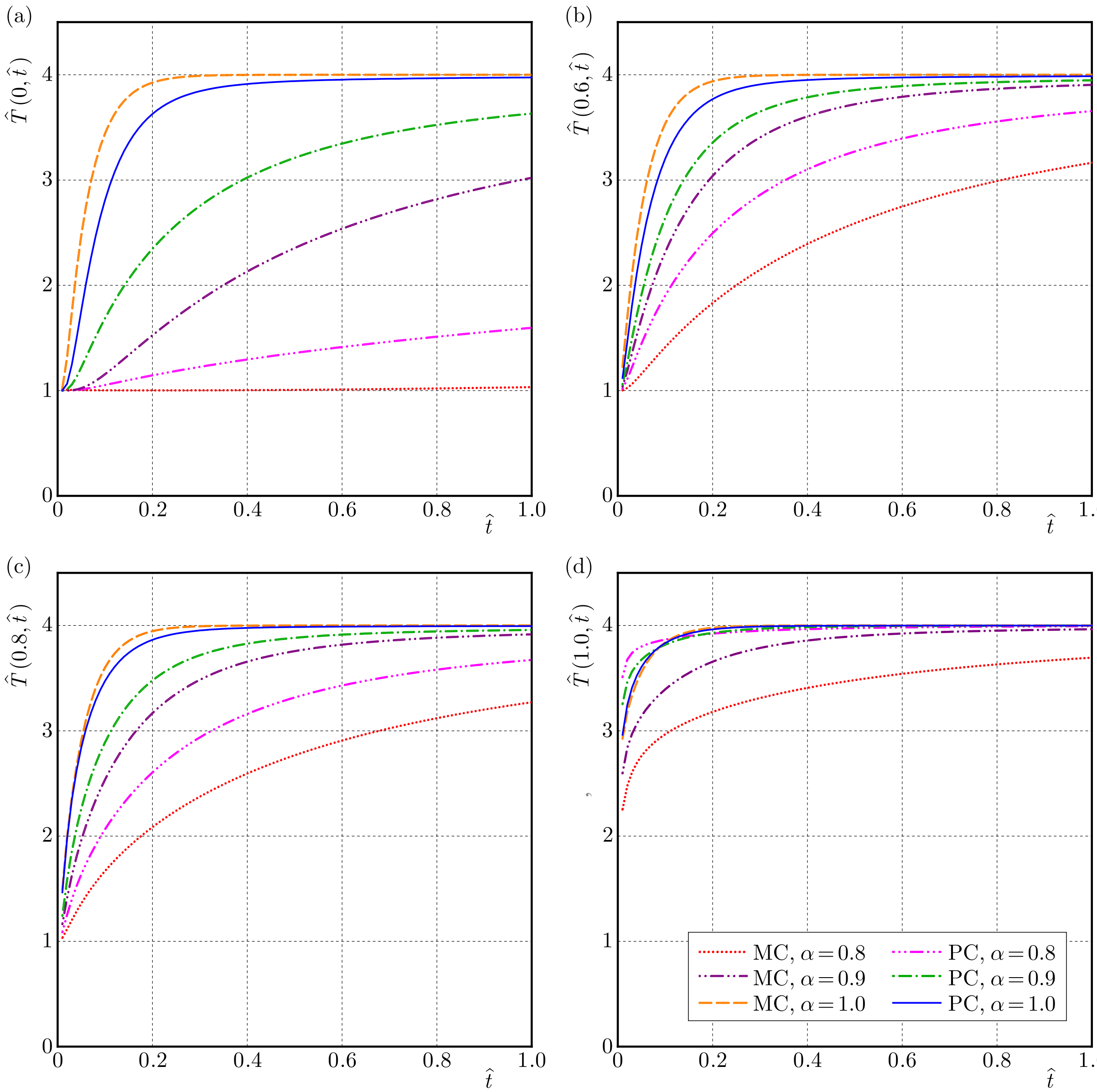

Fig. 2. Non-dimensional temperature $\widehat{T}(\hat{r}, \hat{t})$ as a function of the time $\hat{t}$ in Sphere B for various values of fractional derivatives α and β: (a) $\hat{r} = 0$, (b) $\hat{r} = 0.6$, (c) $\hat{r} = 0.8$, (d) $\hat{r} = 1.0$

5. Conclusions

A solution to the problem of fractional heat conduction in a homogeneous sphere and a composite sphere consisting with a solid sphere and a spherical layer has been presented. The mathematical and physical formulations of the Robin boundary condition and the continuity conditions at the interface have been considered. The temperature distribution in the sphere, under the physical boundary and continuity conditions, has been obtained by using the Laplace transform technique. Numerical results show a significant effect of the order of the Caputo derivative occurring in the heat conduction equation on the time-history of temperature in the sphere. The order of the Riemann-Liouville derivative occurring in the boundary and continuity conditions of the fractional model of heat conduction has a smaller effect on the time-history of temperature in the sphere than the order of the fractional Caputo derivative in the heat conduction equation.

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